249 Replacement: Week 1 Problems

January 22, 2016

Problems from Stanley, Volume I. Notation X [*Y*] means Exercise X in the second edition, corresponding to Y in the first.

1.133a [*1.33*]

Other problems (taken from previous courses by L. Williams and M. Haiman).

1. Rawlings defined an r-*inversion* of a permutation w to be a pair (i, j) of indices with $i < j$ and $0 < w_i - w_j < r$. He also defined an r-*descent* to be an index i for which $w_i \geq w_{i+1} + r$. Finally, he writes $\mathrm{r-maj}(w)$ to denote the number of r-inversions plus the sum of the r-descents. Prove that

$$\sum_{w \in S_n} q^{\mathrm{r-maj}(w)} = (n)_q! = (1)(1+q)(1+q+q^2)\cdots(1+q+q^2+\cdots+q^{n-1}).$$

2. Define $\hat{C}_n(q) = \sum_{w \in \mathcal{C}_n} q^{\mathrm{maj}(w)}$ where $\mathcal{C}_n$ is the set of all *Catalan words* with n zeros and n ones, that is, the set of words in this alphabet such that every initial segment has at least as many 0's as 1's. Prove that

$$\hat{C}_n(q) = \frac{1}{(n+1)_q} \binom{2n}{n}_q.$$

3. (Difficult.) Recall that the *Bell Numbers* B_n count the partitions of the n-element set $[n]$ into any number of nonempty blocks. Recall also that their exponential generating function is

$$\sum_{n \geq 0} \frac{B_n}{n!} x^n = e^{e^x - 1}$$

(and prove this if you don't remember how!)

   - Define the *Bell permutation* of a partition of $[n]$ into blocks $A_1, \ldots, A_k$ as follows. Order $A_1, ..., A_k$ by the size of their maximal elements, and list the elements of each A_i in decreasing order in its position in the ordering. For instance, $\{1, 3, 7\}, \{4, 5\}, \{2, 6\}$ would give rise to the Bell permutation 5462731. Prove that a permutation is a Bell permutation if and only if, whenever a new left-to-right maximum is reached, the sequence decreases until a new left-to-right maximum is reached.
   - Define $B_n(q) = \sum_{\pi \in \mathcal{B}_n} q^{\mathrm{inv}(\pi)}$ where $\mathcal{B}_n$ denotes the set of all Bell permutations. Show that

$$\sum_{n \geq 0} \frac{B_n(q)}{(n)_q!} x^n = \exp_q(\exp_q(x) - 1)$$

where

$$\exp_q(x) = \sum \frac{1}{(n)_q!} x^n.$$

4. Let l divide n. Show that the primitive l-th roots of unity are roots of the polynomial $\binom{n}{k}_q$ in q if and only if l does not divide k.

5. Regarding $\binom{x}{k}$ as a polynomial of degree k in the variable x, prove that a polynomial $f \in \mathbb{Q}[x]$ has the property that $f(n)$ is an integer for all integers n if and only if the coefficients of f with respect to the basis $\{\binom{x}{k} : k \in \mathbb{N}\}$ are integers.

   Hints: Express the coefficients a_k such that $f(x) = \sum_k a_k \binom{x}{k}$ in terms of the iterated differences $(\Delta^m f)(0)$, where $\Delta f(x) = f(x+1) - f(x)$. Use the fact that $0, 1, \ldots, k-1$ are roots of $\binom{x}{k}$, and $\binom{k}{k} = 1$.

6. (a) Show that for each k there is a unique polynomial $Q_k(x)$ of degree k, with coefficients in the field of rational functions $\mathbb{Q}(q)$, such that $Q_k(q^n) = \binom{n}{k}_q$ for all n.

   (b) Prove that a polynomial $f \in \mathbb{Q}(q)[x]$ has the property that $f(q^n) \in \mathbb{Z}[q, q^{-1}]$ for all n if and only if the coefficients of f with respect to the basis $\{Q_k : k \in \mathbb{N}\}$ belong to $\mathbb{Z}[q, q^{-1}]$. Hint: evaluate f at roots of the polynomials $Q_k(x)$.

7. Let $\mathbb{Q}(q)\langle x, y\rangle$ be the algebra of polynomials in non-commuting variables x, y, over the field of rational functions $\mathbb{Q}(q)$, and let $Q_q[x, y] = \mathbb{Q}(q)\langle x, y\rangle / J$, where J is the two-sided ideal generated by $yx - qxy$. Thus $Q_q[x, y]$ is the 'quantum polynomial ring' whose generators satisfy the q-commutation relation $yx = qxy$. Prove the 'quantum q-binomial theorem' that

$$(x+y)^n = \sum_k \binom{n}{k}_q x^k y^{n-k}$$

   holds as an identity in $Q_q[x, y]$.

8. Show that the Stirling numbers of the second kind $S(n, k)$ have a q-analog $S_q(n, k)$ characterized by the following properties:

   (a) They satisfy the recurrence

$$S_q(n, k) = (k)_q S_q(n-1, k) + q^{k-1} S_q(n-1, k-1),$$

   with inital conditions $S_q(0, k) = \delta_{0,k}$ and $S_q(n, 0) = \delta_{n,0}$.

   (b) They satisfy the following q-analog of the classical formula $x^n = \sum_k S(n, k)(x)_k$:

$$((r)_q)^n = \sum_k S_q(n, k)(r)_q(r-1)_q \cdots (r-k+1)_q$$

(c) For each k, they are given by the ordinary generating function

$$\sum_n S_q(n,k)x^n = \frac{q^{\binom{k}{2}}x^k}{(1-x)(1-(2)_q x)\cdots(1-(k)_q x)}.$$

(d) Given a partition $\pi = \{B_1, \dots, B_k\}$ of $[n]$, with the blocks numbered so that $\min(B_i) < \min(B_j)$ for $i < j$, define $\nu(\pi) = \sum_i (i-1)|B_i|$. Then $S_q(n,k) = \sum_\pi q^{\nu(\pi)}$, where the sum is over partitions of $[n]$ into k blocks.

9. (a) Prove that the Eulerian polynomials $A_n(x) = \sum_{\sigma \in S_n} x^{d(\sigma)+1}$ satisfy the recurrence

$$A_n(x) = nxA_{n-1}(x) + x(1-x)A'_{n-1}(x).$$

(b) Prove that the Eulerian polynomials $A_n(x)$ satisfy the following more symmetrical recurrence

$$A_n(x) = xA'_{n-1}(x) + x^n A'_{n-1}(x^{-1}).$$

Use this to compute $A_n(x)$ for $n \leq 5$.

10. One way to define a q-analog of the Eulerian polynomial $A_n(x)$ is

$$A_n(x,q) = \sum_{\sigma \in S_n} x^{d(\sigma)+1} q^{\mathrm{maj}(\sigma)}.$$

(a) Show that with this definition we have

$$\sum_r [r]_q^n x^r = \frac{A_n(x,q)}{(1-x)(1-qx)\cdots(1-q^n x)}$$

(b) Deduce the formula

$$A_n(x,q) = \sum_k [k]_q! S_q(n,k) x^k \prod_{i=k+1}^{n} (1 - xq^i),$$

where $S_q(n,k)$ is the q-analog of a Stirling number.

11. Show that the coefficients $e_{n,d}(q) = \langle x^{d+1} \rangle A_n(x,q)$ (q-analogs of Eulerian numbers) satisfy $e_{n,d}(q) = q^{nd} e_{n,d}(1/q)$ and $e_{n,n-1-d}(q) = q^{\binom{n}{2}} e_{n,d}(1/q)$.

12. Define the descent set of a word $w \in \mathbb{N}^n$ to be $D(w) = \{i \in [n-1] : w(i) > w(i+1)\}$, just as one does for permutations. Similarly, define $\mathrm{maj}(w) = \sum_{d \in D(w)} d$.

(a) Show that if $w \in [r]^n$ and $w(n) = s$, then $w' = (1\ 2\ \cdots\ r)^{r-s} \circ w$ has $\mathrm{maj}(w') = \mathrm{maj}(w) - (k_{s+1} + \cdots + k_r)$, where k_i is the number of occurences of i in the word w.

(b) Use (a) and the following recurrence for q-multinomial coefficients

$$\binom{n}{k_1, k_2, \dots, k_r}_q = \binom{n-1}{k_1 - 1, k_2, \dots, k_r}_q + q^{k_1} \binom{n-1}{k_1, k_2 - 1, \dots, k_r}_q + \cdots + q^{k_1 + \cdots + k_{r-1}} \binom{n-1}{k_1, k_2, \dots, k_r - 1}_q.$$

to prove that

$$\sum_{w \in S_n \cdot (1^{k_1}, 2^{k_2}, \dots, r^{k_r})} t^{\mathrm{maj}(w)} = \sum_{w \in S_n \cdot (1^{k_1}, 2^{k_2}, \dots, r^{k_r})} t^{\mathrm{inv}(w)},$$

for all $k_1 + \cdots + k_r = n$.

(c) Use (b) to prove that for all $D \subseteq [n-1]$,

$$\sum_{\substack{\pi \in S_n \\ D(\pi^{-1}) = D}} t^{\mathrm{maj}(\pi)} = \sum_{\substack{\pi \in S_n \\ D(\pi^{-1}) = D}} t^{\mathrm{inv}(\pi)},$$

that is, inv and maj are equidistributed on inverse descent classes.

(d) Deduce that

$$\sum_{\pi \in S_n} q^{\mathrm{inv}(\pi)} t^{\mathrm{maj}(\pi)} = \sum_{\pi \in S_n} q^{\mathrm{maj}(\pi^{-1})} t^{\mathrm{maj}(\pi)}.$$

Deduce in particular that the left-hand side is symmetric in q and t.

Remark: part (c) implies that $\sum_{\pi \in S_n} x^{d(\pi)+1} q^{\mathrm{maj}(\pi^{-1})} = \sum_{\pi \in S_n} x^{d(\pi)+1} q^{\mathrm{inv}(\pi)}$, which suggests that the common value of these two expressions might be a 'better' q-analog of $A_n(x)$ than the one in the problems above. However, I don't know of nice identities like those above which hold for this alternative q-Eulerian polynomial.

13. Define the q-*derivative* $(d/dx)_q$ by

$$(d/dx)_q f(x) = \frac{f(x) - f(qx)}{x(1-q)},$$

so that $(d/dx)_q x^n = [n]_q x^{n-1}$, for example. Note that if f is a polynomial, the numerator vanishes both at $x = 0$ and $q = 1$, so it is divisible by the denominaor.

(a) Verify the product rule for q-derivatives

$$(d/dx)_q f(x) g(x) = ((d/dx)_q f(x)) \cdot g(x) + f(qx) \cdot (d/dx)_q g(x).$$

(b) Show that the q-Eulerian polynomials $A_n(x, q)$ defined above satisfy the following q-analog of the recurrence above:

$$A_n(x, q) = x (d/dx)_q A_{n-1}(x, q) + q^{\binom{n}{2}} x^n \left((d/dx)_q A_{n-1}(x, q) \right)_{x \mapsto x^{-1},\ q \mapsto q^{-1}}.$$