Homework 10 Solutions

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1 5.21

(a) These events are clearly independent, so

$$\Pr(\text{both are heads}) = \Pr(\text{the first is heads})\Pr(\text{the last is heads}) = \frac{1}{4}$$

(b)

$$\Pr(\text{the first or last is heads}) = 1 - \Pr(\text{both are tails}) = \frac{3}{4}$$

(c) It's 1/2

(d)

$$\Pr(\text{exactly } k \text{ heads}) = \frac{\binom{k}{10}}{2^{10}}$$

You were asked to evaluate this for each $0 \leq k \leq 10$.

(e) By the previous part,

$$\Pr(\text{even number of heads}) = \sum_{i=0}^{5} \frac{\binom{2i}{10}}{2^{10}}$$

(f) Here's one answer:

$$\Pr(\text{odd number of heads}) = 1 - \Pr(\text{even number of heads})$$

Plugging in the answer from (e) gives the right number.

2 5.25

(a) This one is a bit cheeky. You might think that we would have to compute some conditional probabilities. We actually don't have to compute anything: the answer is just 1/3, the same as if we were the first person to pick. To see this, it might help to imagine that the coins are stacked randomly in a tower at the beginning. No matter what the other people do, we are going to get the 6th coin from the top. This coin, just like every other coin, has a 1/3 chance of being gold.

This reminds me of a puzzle. Suppose you play the following game with a friend: they shuffle a deck of cards and reveal them one at a time. At any time, you can tell them to stop, and then if the next card is red, you win, and if it is black, you lose. What is the strategy that maximizes your chances of winning?

(b) As in part (a), it doesn't matter what order the coins are drawn in. So, the answer is the same as the probability that 5 people in a row draw silver coins from an urn containing 9 gold coins and 20 silver coins. This is

$$\frac{20}{29} \cdot \frac{19}{29} \cdot \frac{18}{29} \cdot \frac{17}{29} \cdot \frac{16}{29}$$

3 5.28

We proceed as in the example. Let E be the event "m does not have property A", and let F be the event "the algorithm fails N times in a row". We will use Bayes's formula:

$$\Pr(E|F) = \frac{\Pr(F|E)\Pr(E)}{\Pr(F|E)\Pr(E) + \Pr(F|E^c)\Pr(E^c)}$$

We compute the pieces of the right hand side. Assumption (1) implies

$$\Pr(F|E) = 1$$

This makes sense: if m doesn't have property A, then the algorithm has to fail every time (otherwise it would be lying to us). By definition,

$$\Pr(E) = \delta$$

Hence,

$$\Pr(E^c) = 1 - \delta$$

Finally, we have

$$\begin{aligned}
\Pr(F|E^c) &= \Pr(\text{the algorithm fails } N \text{ times in a row } |m \text{ has property } A) \\
&= (\Pr(\text{the algorithm fails } |m \text{ has property } A)^N \\
&= (1 - \Pr(\text{the algorithm returns "YES" } |m \text{ has property } A))^N \\
&\leq (1-p)^N
\end{aligned}$$

Putting this together, we get

$$\Pr(E|F) \geq \frac{\delta}{\delta + (1-p)^N(1-\delta)}$$