Homework 8

(1) Do exercise 3 from section 15.4.

(2) Here is a group which you should know about, but which we haven't discussed yet. It is called the quaternion group, and is described in Example 3.15 of the book. As a set, we have

$$Q_8 = \{\pm 1, \pm I, \pm J, \pm K\}$$

where 1 denotes the 2x2 identity matrix and I, J, K are certain 2x2 matrices. You don't actually have to think about matrices to work with this group: all you need to know is that they satisfy the relations

$$I^2 = J^2 = K^2 = IJK = -1$$

Find all subgroups of Q_8 (there are three of order 4 and one of order 2). Show that every subgroup of Q_8 is normal (despite the fact that Q_8 is nonabelian!).

(3) In this exercise we are going to classify all groups of order 8. We already know five of them:

$$D_8, \quad Q_8, \quad \mathbb{Z}_8, \quad \mathbb{Z}_4 \times \mathbb{Z}_2, \quad \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$$

(a) Show that none of these are isomorphic to any other (for each one, find some properties which only that group has among the five).

(b) You are now going to show that the five groups above are a complete list of the groups of order 8. Let G be a group of order 8. We consider the possibilities for the orders of elements of G.

Case 1: If there is an element in G whose order is 8, then $G \cong \mathbb{Z}_8$ (you don't need to say anything here).

Case 2: Suppose that every nonidentity element of G has order 2. Show that $G \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

Case 3: Suppose that we are not in Case 1 or 2. In particular, there is an element $g \in G$ whose order is 4. Let $H = \langle g \rangle \leq G$ be the cyclic subgroup generated by g. We know that H has to be normal (as $|H| = |G|/2$). We consider the following subcases:

Case 3a: There is an element $x \in G \setminus H$ of order 2, and $K = \langle x \rangle$ is normal. In this case, show that $G \cong \mathbb{Z}_4 \times \mathbb{Z}_2$. **Hint:** use the recognition theorem for direct products.

Case 3b: There is an element $x \in G \setminus H$ of order 2, and $K = \langle x \rangle$ is NOT normal. In this case, show that $G \cong D_4$. **Hint:** try to show that $gx = xg^{-1}$. To do this, first show that either $xgx^{-1} = g$ or $xgx^{-1} = g^{-1}$. To rule out the first case, note that if $xgx^{-1} = g$, then $gxg^{-1} = x$. Then show that this implies $N_G(K) = G$, which contradicts our assumption that K is not normal.

Case 3c: Every element of $G \setminus H$ has order 4. In this case, show that $G \cong Q_8$. Here are a few steps to get you started on the right track. First, fix an element $h \in G \setminus H$. Show that the elements

$$1, g, g^2, g^3, h, gh, g^2h, g^3h$$

are all distinct. So, this is a complete list of the elements of G. Now, find an isomorphism $G \to Q_8$ which satisfies $g \mapsto I$, $h \mapsto J$, $gh \mapsto K$ (remember, by assumption the last four elements in the above list have order 4).

(4) The goal of this exercise is to give a different proof that A_5 is simple, using Sylow's theorems. Note that $|A_5| = 60 = 2^2 \cdot 3 \cdot 5$.

(a) Show that $n_3 = 10$ and $n_5 = 6$. (**Hint:** you should be able to exhibit explicitly all 10 Sylow 3-subgroups and all 6 Sylow 5-subgroups. Then use Sylow's theorems to argue that you have found all of them.)

(b) Write down two Sylow 2-subgroups of A_5 which have trivial intersection. Deduce in particular that $n_2 \geq 2$. (**Hint:** look at the order 4 subgroup of A_4 which I wrote down in the lectures.)

(c) Suppose that $N \subset A_5$ is a normal subgroup. Suppose that 3 divides $|N|$. Prove that every Sylow 3-subgroup of A_5 is contained in N. Similarly, show that if 5 divides $|N|$, then every Sylow 5-subgroup of A_5 is contained in N.

(d) Suppose that either 3 or 5 divides $|N|$. Using part (c) and Lagrange's theorem, deduce that in fact both 3 and 5 divide $|N|$. Conclude that $N = A_5$. (**Hint:** note that any two subgroups of N each of which have order 3 or 5 must have trivial intersection.)

(e) Using Sylow's theorems and part (b), show that A_5 does not have a normal subgroup of order 4.

(f) Show that if N is a normal subgroup of A_5 of order 2, then N is contained in every Sylow 2-subgroup of A_5. Using part (b), deduce that A_5 does not have a normal subgroup of order 2.

(g) Using Lagrange's theorem, deduce that A_5 is simple.