Homework 7

(4) Here is a new group we haven't seen before. It is called the "Frobenius group" of order 20, and has presentation

$$F_{20} = \langle x, y | x^4 = y^5 = 1, xyx^{-1} = y^2 \rangle$$

This group can be constructed as a semidirect product. Show explicitly (by writing them down) that there are exactly ~~three~~ **four** homomorphisms $\varphi : \mathbb{Z}/4 \to \mathrm{Aut}(\mathbb{Z}/5)$. Write down a presentation for each of the resulting semidirect products $\mathbb{Z}/5 \rtimes_\varphi \mathbb{Z}/4$. One of these presentations should be exactly the presentation of F_{20} above. Show that **two** of these four groups are isomorphic (use question 3). Finally, show that none of the other groups are isomorphic to each other or to these two, and therefore that we end up with **three** distinct groups of order 20 up to isomorphism.

Solution: We have $\mathrm{Aut}(\mathbb{Z}/5) = (\mathbb{Z}/5)^\times$. As a set, this is $\{1, 2, 3, 4\}$, with the group operation being multiplication modulo 5. We want to determine all homomorphisms $\mathbb{Z}/4 \to \mathrm{Aut}(\mathbb{Z}/5)$. Later in the problem it will be helpful to think of $\mathrm{Aut}(\mathbb{Z}/5)$ as $(\mathbb{Z}/5)^\times$, but for now lets just think about it abstractly as a group. We know that it isomorphic as a group to the cyclic group with four elements (by a theorem from lecture, for instance). An isomorphism is given by sending the generator of $\mathbb{Z}/4$ to 2, which has order 4 in $(\mathbb{Z}/5)^\times$. Thus, we need to determine all homomorphisms $\mathbb{Z}/4 \to \mathbb{Z}/4$. Here is why there are four of these: first, there is the trivial homomorphism. There are also two injective homomorphisms, given by the two elements in $\mathrm{Aut}(\mathbb{Z}/4) = (\mathbb{Z}/4)^\times = \{1, 3\}$. Finally, there is a unique nontrivial, non-injective homomorphism, corresponding to modding out by the unique proper nontrivial subgroup of $\mathbb{Z}/4$, which is $\mathbb{Z}/2 = \{0, 2\}$.

Now, lets return to using $\mathrm{Aut}(\mathbb{Z}/5) = (\mathbb{Z}/5)^\times = \{1, 2, 3, 4\}$. The homomorphisms we determined above are then

$\varphi_1 : \mathbb{Z}/4 \to \mathrm{Aut}(\mathbb{Z}/5)$	$\varphi_2 : \mathbb{Z}/4 \to \mathrm{Aut}(\mathbb{Z}/5)$	$\varphi_3 : \mathbb{Z}/4 \to \mathrm{Aut}(\mathbb{Z}/5)$	$\varphi_4 : \mathbb{Z}/4 \to \mathrm{Aut}(\mathbb{Z}/5)$
$0 \mapsto 1$	$0 \mapsto 1$	$0 \mapsto 1$	$0 \mapsto 1$
$1 \mapsto 1$	$1 \mapsto 2$	$1 \mapsto 3$	$1 \mapsto 4$
$2 \mapsto 1$	$2 \mapsto 4$	$2 \mapsto 4$	$2 \mapsto 1$
$3 \mapsto 1$	$3 \mapsto 3$	$3 \mapsto 2$	$3 \mapsto 4$

Here, φ_1 is the trivial homomorphism, which sends everything to the identity automorphism of $\mathbb{Z}/5$, which is multiplication by 1. The second homomorphism φ_2 is determined by choosing to send 1 to 2, which forces 2 to go to $2 * 2 = 4$ and 3 to $2 * 2 * 2 = 8 \equiv 3 \pmod 5$, etc.

Remembering the discussion of presentations of semidirect products from lecture, we get the following presentations.

$$\mathbb{Z}/5 \rtimes_{\varphi_1} \mathbb{Z}/4 = \langle x, y | x^4 = y^5 = 1, xyx^{-1} = y \rangle$$
$$\mathbb{Z}/5 \rtimes_{\varphi_2} \mathbb{Z}/4 = \langle x, y | x^4 = y^5 = 1, xyx^{-1} = y^2 \rangle$$
$$\mathbb{Z}/5 \rtimes_{\varphi_3} \mathbb{Z}/4 = \langle x, y | x^4 = y^5 = 1, xyx^{-1} = y^3 \rangle$$
$$\mathbb{Z}/5 \rtimes_{\varphi_4} \mathbb{Z}/4 = \langle x, y | x^4 = y^5 = 1, xyx^{-1} = y^4 \rangle$$

Note that $\mathbb{Z}/5 \rtimes_{\varphi_1} \mathbb{Z}/4$ is just the direct product, because the relation $xyx^{-1} = y$ is equivalent to $xy = yx$. Also, $\mathbb{Z}/5 \rtimes_{\varphi_2} \mathbb{Z}/4$ is exactly the above presentation of F_{20}.

We will show that $\mathbb{Z}/5 \rtimes_{\varphi_2} \mathbb{Z}/4 \cong \mathbb{Z}/5 \rtimes_{\varphi_3} \mathbb{Z}/4$. To see this, we'll use problem 3b. Consider the automorphism ψ of $K = \mathbb{Z}/4$ given by multiplication by 3 modulo 4. This is the unique nontrivial automorphism of $\mathbb{Z}/4$, and it has order 2 because $3 * 3 = 9 \equiv 1 \pmod 4$. We have

$$\begin{array}{rccccc} \varphi_2 \circ \psi: & \mathbb{Z}/4 & \xrightarrow{\psi} & \mathbb{Z}/4 & \xrightarrow{\varphi_2} & \mathrm{Aut}(\mathbb{Z}/5) \\ & 0 & \mapsto & 0 & \mapsto & 1 \\ & 1 & \mapsto & 3 & \mapsto & 3 \\ & 2 & \mapsto & 2 & \mapsto & 4 \\ & 3 & \mapsto & 1 & \mapsto & 2 \end{array}$$

Thus, $\varphi_2 \circ \psi = \varphi_3$. Note that ψ has order 2, so $\psi = \psi^{-1}$. Hence, problem 3b tells us that the function

$$\mathbb{Z}/5 \rtimes_{\varphi_2} \mathbb{Z}/4 \cong \mathbb{Z}/5 \rtimes_{\varphi_3} \mathbb{Z}/4$$

defined by $(y^i, x^j) \mapsto (y^i, x^{3j})$ is an isomorphism of groups. In terms of the above presentations, this is the homomorphism

$$\begin{array}{rcl} \langle x, y | x^4 = y^5 = 1, xyx^{-1} = y^2 \rangle & \to & \langle x, y | x^4 = y^5 = 1, xyx^{-1} = y^3 \rangle \\ x & \mapsto & x^3 \\ y & \mapsto & y \end{array}$$

The last thing to do is to show that $\mathbb{Z}/5 \rtimes_{\varphi_1} \mathbb{Z}/4$, $\mathbb{Z}/5 \rtimes_{\varphi_2} \mathbb{Z}/4$, and $\mathbb{Z}/5 \rtimes_{\varphi_4} \mathbb{Z}/4$ are *not* isomorphic. The first group is abelian, and the other two are not (for instance, in the last two groups $xy \neq yx$). So, it remains to show that $\mathbb{Z}/5 \rtimes_{\varphi_2} \mathbb{Z}/4$ is not isomorphic to $\mathbb{Z}/5 \rtimes_{\varphi_4} \mathbb{Z}/4$. One way to see this is by looking at the centers of these groups. On the one hand, we compute that in $\mathbb{Z}/5 \rtimes_{\varphi_2} \mathbb{Z}/4$ we have

$$x(x^i y^j)x^{-1} = x^i y^{2j} \qquad y(x^i y^j)y^{-1} = x^i y^{3^i + j - 1}$$

If $x^i y^j$ is in the center, then by the first relation, we would have $j \equiv 2j \pmod 5$, so $j \equiv 0 \pmod 5$. By the second, we have $3^i \equiv 1 \pmod 5$, which implies $i \equiv 0 \pmod 4$. So, $x^i y^j = 1$. Therefore

$$\mathbf{Z}(\mathbb{Z}/5 \rtimes_{\varphi_2} \mathbb{Z}/4) = 1$$

On the other hand, in $\mathbb{Z}/5 \rtimes_{\varphi_4} \mathbb{Z}/4$ we have

$$x(x^i y^j)x^{-1} = x^i y^{4j} \qquad y(x^i y^j)y^{-1} = x^i y^{4^i + j - 1}$$

If $x^i y^j$ is in the center, then by the first relation, we would have $j \equiv 4j \pmod 5$, so $j \equiv 0 \pmod 5$. By the second, we have $4^i \equiv 1 \pmod 5$, which implies $i \equiv 2 \pmod 4$. Therefore

$$\mathbf{Z}(\mathbb{Z}/5 \rtimes_{\varphi_4} \mathbb{Z}/4) = \{1, x^2\}$$

We conclude that $\mathbb{Z}/5 \rtimes_{\varphi_2} \mathbb{Z}/4$ and $\mathbb{Z}/5 \rtimes_{\varphi_4} \mathbb{Z}/4$ are not isomorphic.