Homework 7

(1) Let G be a finite group. Suppose that $|G| = p^a q^b$ for two primes p and q, and that $n_p(G) = 1$. Prove that G is a semidirect product of two smaller groups.

Edit: The first version of this problem omitted the assumption that $|G|$ is divisible by only two distinct primes. The result is still true without this assumption, but the proof was harder than I intended. The most general statement along these lines that I know of is the "Schur–Zassenhaus theorem":

Theorem 1 (Schur-Zassenhaus) *Let G be a finite group and let H be a normal subgroup of G such that the orders of H and G/H are coprime. There exists a subgroup K of G such that $G = H \rtimes K$.*

(2) Show that A_4 is isomorphic to a semidirect product of two smaller groups. Write down the corresponding presentation for A_4. It might be helpful to look at the discussion in lecture about presentations of semidirect products.

(3) Let H and K be groups and let $\varphi : K \to \mathrm{Aut}(H)$ be a homomorphism. As we've seen in lecture, it can happen that for for some other homomorphism φ' we have an isomorphism

$$H \rtimes_\varphi K \cong H \rtimes_{\varphi'} K$$

between the semidirect products. In this exercise, we will identify two general situations in which this happens.

Caution: these conditions are not sufficient. That is, it can happen that neither of these hold, but still the semidirect products are isomorphic.

(a) Let $\sigma \in \mathrm{Aut}(H)$ be a fixed automorphism. Define $\varphi' : K \to \mathrm{Aut}(H)$ by $\varphi'_k = \sigma \circ \varphi_k \circ \sigma^{-1}$. Show that the function

$$H \rtimes_\varphi K \to H \rtimes_{\varphi'} K$$

defined by $(h, k) \mapsto (\sigma(h), k)$ is an isomorphism of groups.

(b) Let $\psi \in \mathrm{Aut}(K)$ be an automorphism of K. Given a homomorphism $\varphi : K \to \mathrm{Aut}(H)$, we can precompose with ψ to get a new homomorphism $\varphi' = \varphi \circ \psi : K \to \mathrm{Aut}(H)$. In other words, $\varphi'_k = \varphi_{\psi(k)}$. Show that the function

$$H \rtimes_\varphi K \cong H \rtimes_{\varphi'} K$$

defined by $(h, k) \mapsto (h, \psi^{-1}(k))$ is an isomorphism of groups, where $\psi^{-1} : K \to K$ is the inverse of ψ.

(4) Here is a new group we haven't seen before. It is called the "Frobenius group" of order 20, and has presentation

$$F_{20} = \langle x, y | x^4 = y^5 = 1, xyx^{-1} = y^2 \rangle$$

This group can be constructed as a semidirect product. Show explicitly (by writing them down) that there are exactly ~~three~~ **four** homomorphisms $\varphi : \mathbb{Z}/4 \to \mathrm{Aut}(\mathbb{Z}/5)$. Write down a presentation for each of the resulting semidirect products $\mathbb{Z}/5 \rtimes_\varphi \mathbb{Z}/4$. One of these presentations should be exactly the presentation of

F_{20} above. Show that **two** of these four groups are isomorphic (use question 3). Finally, show that none of the other groups are isomorphic to each other or to these two, and therefore that we end up with **three** distinct groups of order 20 up to isomorphism.

(5) In this exercise you are going to classify all groups of order 20. We will show that every group of order 20 is isomorphic to one of the following

$$\mathbb{Z}/5 \times \mathbb{Z}/2 \times \mathbb{Z}/2, \quad \mathbb{Z}/5 \times \mathbb{Z}/4, \quad \mathbb{Z}/5 \rtimes \mathbb{Z}/4, \quad F_{20}, \quad D_{20}$$

Here, the implicit homomorphism φ in the third group is the unique *nontrivial, non-injective* homomorphism.

(a) Let G be a group of order 20. Show that $n_5 = 1$. Deduce that $G \cong \mathbb{Z}/5\mathbb{Z} \rtimes_\varphi P$, where P is a Sylow 2-subgroup of G (so P is a group of order 4).

Case 1: Suppose that $P \cong \mathbb{Z}/4\mathbb{Z}$. In problem (4), you showed that there are exactly ~~three~~ **four** homomorphisms $\mathbb{Z}/4 \to \mathrm{Aut}(\mathbb{Z}/5)$. Using problem (3), show that these homomorphisms give rise to three distinct groups in the above list.

Case 2: Suppose that $P \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Taking φ to be trivial gives the direct product (the first group in the list). Show that there are exactly 3 nontrivial homomorphisms $\varphi : \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \to \mathrm{Aut}(\mathbb{Z}/5\mathbb{Z})$. Using Problem (3b) and Problem (2) from Homework 6, show that all the resulting semidirect products are isomorphic. Prove that the resulting semidirect product is isomorphic to D_{20}.

(b) Show that no two of these groups are isomorphic to each other.

(6) How many abelian groups of order $540 = 2^2 \cdot 3^3 \cdot 5$ are there, up to isomorphism? Write them down using the invariant factor decomposition.

(7) Finish the proof of the fundamental theorem of finite abelian groups from Lecture 17.1 by proving the following:

Theorem 2 *Let p be a prime and let n be a positive integer. Let $\beta = (\beta_1, \ldots, \beta_i)$ and $\alpha = (\alpha_1, \ldots, \alpha_j)$ be two partitions of n (not necessarily of the same length). Let G_β and G_α be the corresponding abelian p-groups of order n. If $G_\beta \cong G_\alpha$, then the partitions β and α are equal.*

Hint: a good way to do this is to use "Claim 2" from Lecture 17.2 repeatedly.