Homework 6

(1) Write down an isomorphism $\mathrm{Aut}(\mathbb{Z}/8\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. **Hint**: note that by part (1), the negation map ν gives an element of $\mathrm{Aut}(\mathbb{Z}/8\mathbb{Z})$ of order 2. So, to find all of the automorphisms, you just need one more.

(2) Write down an isomorphism $\mathrm{Aut}(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}) \cong S_3$.

(3) Here is a group which you should know about, but which we haven't discussed yet. It is called the quaternion group Q_8, and has presentation

$$Q_8 = \langle i, j, k | i^2 = j^2 = k^2 = ijk \rangle$$

It has order 8. You can read about it in Section 1.5 of the book. When talking about the quaternion group, it is usual to denote the element $i^2 = j^2 = k^2$ by the symbol "-1". The book uses this notation.

Show that Q_8 is not a semidirect product of any two nontrivial groups. You might want to use the lattice of subgroups for Q_8 on page 69.

(4) In this exercise we are going to classify all groups of order 8. We already know five of them:

$$D_8, \quad Q_8, \quad \mathbb{Z}/8\mathbb{Z}, \quad \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}, \quad \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$$

(a) Show that none of these are isomorphic to any other (this should be quick: for each one, find some properties which only that group has among the five).

(b) You are now going to show that the five groups above are a complete list of the groups of order 8. Let G be a group of order 8. We consider the possibilities for the orders of elements of G.

Case 1: If there is an element in G whose order is 8, then $G \cong \mathbb{Z}/8\mathbb{Z}$ (you don't need to say anything here).

Case 2: Suppose that every nonidentity element of G has order 2. Show that $G \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

Case 3: Suppose that we are not in Case 1 or 2. In particular, there is an element $g \in G$ whose order is 4. Let $H = \langle g \rangle \leq G$ be the cyclic subgroup generated by g. Consider the complement $G \setminus H$ of H (that is, all the elements in G that are not in H). We consider two subcases:

Case 3a: There is an element $x \in G \setminus H$ of order 2. In this case, show that either $G \cong D_8$ or $G \cong \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. **Hint:** note that the subgroup $H \leq G$ has index two, so it is normal. Consider the subgroup $K = \langle x \rangle \leq G$. Using our recognition theorem for semidirect products, deduce that $G = H \rtimes_\varphi K$, and consider the possibilities for φ.

Case 3b: Every element of $G \setminus H$ has order 4. In this case, show that $G \cong Q_8$. Here are a few steps to get you started on the right track. First, fix an element $h \in G \setminus H$. Show that the elements

$$1, g, g^2, g^3, h, gh, g^2h, g^3h$$

are all distinct. So, this is a complete list of the elements of G. Now, find an isomorphism $G \to Q_8$ which satisfies $g \mapsto i$, $h \mapsto j$, $gh \mapsto k$ (remember, by assumption the last four elements in the above list have order 4).

(5) Complete the classification of groups of order pq from the lecture. Suppose that $p < q$ are primes and that $p | q - 1$. Let

$$\varphi, \nu : \mathbb{Z}/p\mathbb{Z} \to \mathrm{Aut}(\mathbb{Z}/q\mathbb{Z})$$

be two homomorphisms neither of which are trivial. Construct an isomorphism

$$\mathbb{Z}/q\mathbb{Z} \rtimes_\varphi \mathbb{Z}/p\mathbb{Z} \cong \mathbb{Z}/q\mathbb{Z} \rtimes_\nu \mathbb{Z}/p\mathbb{Z}$$

As in the lectures, lets use multiplicative notation, and let k be a generator for $\mathbb{Z}/p\mathbb{Z}$ and h be a generator for $\mathbb{Z}/q\mathbb{Z}$. Because $\mathbb{Z}/p\mathbb{Z}$ is cyclic, φ and ν are determined by where they send k. Using the fact that $\mathrm{Aut}(\mathbb{Z}/q\mathbb{Z})$ is cyclic of order $q-1$, show that there is an integer m such that the homomorphisms φ and ν^m are equal. Now show that $(h, k) \mapsto (h, k^m)$ defines a homomorphism. This problem is a special case of exercise 6 in section 5.5 of the book.