Homework 5

(1) The goal of this exercise is to prove the orbit–stabilizer theorem, which is the following:

Theorem 0.1 Let G be a finite group which acts on a finite set A. For any $x \in A$, we have

$$|\operatorname{Orb}(x)| = |G|/|\operatorname{Stab}(x)|$$

Furthermore, if $x_1, \ldots, x_r \in A$ is a list of representatives of the orbits the action which have size strictly greater than 1, then we have

$$|A| = |\operatorname{Fix}_G(A)| + \sum_{i=1}^{r} [G : \operatorname{Stab}(x_i)]$$

Remember, here $\operatorname{Orb}(x) = \{a \in A | g \cdot a = x \text{ for some } g \in G\}$ is the *orbit* of x, $\operatorname{Stab}(x) = \{g \in G | g \cdot x = x\}$ is the *stabilizer* of x, and $\operatorname{Fix}_G(A) = \{x \in A | g \cdot x = x \text{ for all } g \in G\}$ is the set of *fixed points* of the action. Equivalently, $\operatorname{Fix}_G(A)$ is the set of $x \in A$ such that $|\operatorname{Orb}(x)| = 1$. Note that this theorem recovers the class equation as a special case (consider the action of G on itself by conjugation).

(a) Define a function $f : G \to \operatorname{Orb}(x)$ by $f(g) = g \cdot x$. Show that the function

$$\overline{f} : G/\operatorname{Stab}(x) \to \operatorname{Orb}(x)$$

given by $\overline{f}(g \operatorname{Stab}(x)) = g \cdot x$ is well defined. (Remember, $G/\operatorname{Stab}(x)$ denotes the set of left cosets of $\operatorname{Stab}(x)$ in G. We do not know that $\operatorname{Stab}(x)$ is a normal subgroup of G, so $G/\operatorname{Stab}(x)$ is not a group).

(b) Show that $\overline{f}$ is a bijection, and prove that $|\operatorname{Orb}(x)| = |G|/|\operatorname{Stab}(x)|$.

(c) Finish the proof of the orbit stabilizer theorem.

(2) Using the orbit stabilizer theorem, prove the fixed point formula from Lecture 9, which is the following statement:

Theorem 0.2 Let p be a prime number. If G is a p-group acting on a finite set A, then

$$|A| \equiv |\operatorname{Fix}_G(A)| \pmod{p}$$

(**Hint:** show that for any $x \in A$, either $|\operatorname{Orb}(x)| = 1$, or p divides $|\operatorname{Orb}(x)|$).

(4) Show that any group of order 40 or 45 cannot be simple. (**Hint:** in either case, show that $n_p = 1$ for some prime p. Then show that in this case the unique Sylow p-subgroup is normal).

(3) The goal of this exercise is to prove that A_5 is simple. Remember, this means that A_5 doesn't have any normal subgroups except for 1 and all of A_5. Note that $|A_5| = 60 = 2^2 \cdot 3 \cdot 5$.

(a) Show that $n_3 = 10$ and $n_5 = 6$. (**Hint:** you should be able to exhibit explicitly all 10 Sylow 3-subgroups and all 6 Sylow 5-subgroups. Then use Sylow's theorems to argue that you have found all of them.)

(b) Write down two Sylow 2-subgroups of A_5 which have trivial intersection. Deduce in particular that $n_2 \geq 2$. (**Hint:** look at the order 4 subgroup of A_4 which I wrote down in the lectures.)

(c) Suppose that $N \trianglelefteq A_5$ is a normal subgroup. Suppose that 3 divides $|N|$. Prove that every Sylow 3-subgroup of A_5 is contained in N. Similarly, show that if 5 divides $|N|$, then every Sylow 5-subgroup of A_5 is contained in N.

(d) Suppose that either 3 or 5 divides $|N|$. Using part (c) and Lagrange's theorem, deduce that in fact both 3 and 5 divide $|N|$. Conclude that $N = A_5$. (**Hint:** note that any two subgroups of N each of which have order 3 or 5 must have trivial intersection.)

(e) Using Sylow's theorems and part (b), show that A_5 does not have a normal subgroup of order 4.

(f) Show that if N is a normal subgroup of A_5 of order 2, then N is contained in every Sylow 2-subgroup of A_5. Using part (b), deduce that A_5 does not have a normal subgroup of order 2.

(g) Using Lagrange's theorem, deduce that A_5 is simple.