Homework 4

Section 4.2, problem (10) Let G be a nonabelian group of order 6. Let's first show that there must be an element of order 2 in G. The possible orders of non-identity elements of G are 2,3, and 6. If there was an element of order 6, then our group would be cyclic, hence abelian, which it is not by assumption. So there are no elements of order 6. Suppose that all non-identity elements of G had order 3. Then, every non-identity element would generate a cyclic subgroup of order 3. Any two distinct such subgroups must intersect only in the identity. Thus, any group such that every non-identity element has order 3 must have size of the form $1 + 2 + 2 + \cdots + 2$ for some number of 2's. But 6 cannot be written in this way. So, we conclude that there must be at least one element of order 2 in G.

Let $\sigma \in G$ be an element of order 2. Consider the cyclic subgroup $H = \langle \sigma \rangle \leq G$. If this subgroup were normal, then we would have $g\sigma g^{-1} \in H$ for all $g \in G$. But the only non-identity element of H is σ, so we would actually have $g\sigma g^{-1} = \sigma$ for all $g \in G$. But this means that $\sigma \in Z(G)$. So, 2 divides $|Z(G)|$. As G is not abelian, we must have $2 = |Z(G)|$. But then the quotient $G/Z(G)$ has order 3, hence is cyclic. By an exercise on homework 3, this implies that G is abelian, a contradiction. We conclude that H is not normal.

We now finish the problem. Consider the action of G on the set of left cosets G/H given by left multiplication. There are 3 such cosets, so this corresponds to a homomorphism $\varphi : G \to S_3$. We claim that φ is injective. Suppose that $g \in G$ is in $\ker \varphi$. This means that $gg'H = g'H$ for all $g' \in G$. Equivalently, this means that $g \in g'Hg'^{-1}$ for all $g' \in G$. Consider some g' such that $g'Hg'^{-1} \neq H$ (we showed such a g' exists in the last paragraph). We then have $g'Hg'^{-1} \cap H = 1$ (this is the intersection of two subgroups of order 2 which are not equal). We conclude that $g = 1$. Thus, φ is injective, and because both groups have order 6, it is also surjective, and hence an isomorphism.

We now give the classification of groups of order 6. We claim that there are only two, namely S_3 and $\mathbf{Z}/6\mathbf{Z}$. Indeed, we have just shown that if G is a nonabelian group of order 6, then $G \cong S_3$. We claim that if G is an abelian group of order 6 then $G \cong \mathbf{Z}/6\mathbf{Z}$. To see this, it will suffice to produce an element of order 6 in G. One way to do this is to first produce an elements of order 2, say x, and also an element of order 3, say y. The element xy then has order 6.