For a vector \( \mathbf{x} = (x_1, \ldots, x_n) \), the function

\[
f(\mathbf{x}) = |\mathbf{x}| = \sqrt{\mathbf{x} \cdot \mathbf{x}} = \sqrt{x_1^2 + \ldots + x_n^2}
\]

can be expressed as the composition of a (positive) polynomial and a square root. Since the square root is continuous on positive numbers and polynomials are always continuous we see that \( f \) is the composition of continuous functions. But the composition of continuous functions is continuous.

Alternate proof:

#14.3 95:

a) Away from \((0, 0)\) we may take partial derivatives as usual, so

\[
f_x = \frac{y(x^4 + 4x^2y^2 - y^4)}{(x^2 + y^2)^2}
\]

and

\[
f_y = \frac{x(x^4 - 4x^2y^2 - y^4)}{(x^2 + y^2)^2}
\]

c) Using the limit definition of the derivative,

\[
f_x(0, 0) = \lim_{h \to 0} \frac{f(h, 0) - f(0, 0)}{h} = \lim_{h \to 0} \frac{h^3(0) - h(0)^3}{h^2 + 0^2} = 0
\]
and
\[
f_y(0, 0) = \lim_{h \to 0} \frac{f(0, h) - f(0, 0)}{h} = \lim_{h \to 0} \frac{(0)^3h - 0h^3}{0^2 + h^2} = 0.
\]
Notice that this with part b) shows that \( f_x \) and \( f_y \) are continuous.

d) Let us use the limit definition again, as the equations will greatly simplify at \((0, 0)\):
\[
f_{xy}(0, 0) = \lim_{h \to 0} \frac{f_x(0, h) - f_x(0, 0)}{h} = \lim_{h \to 0} \frac{h(0^4 + 40^2h^2 - h^4)}{h(0^2 + h^2)^2} = \lim_{h \to 0} \frac{-h^5}{h^5} = -1
\]
and
\[
f_{yx}(0, 0) = \lim_{h \to 0} \frac{f_y(h, 0) - f_y(0, 0)}{h} = \lim_{h \to 0} \frac{h(0^4 - 4^20^2 - 0^4)}{h(h^2 + 0^2)^2} = \lim_{h \to 0} \frac{h^5}{h^5} = 1.
\]
e) To use Clairaut’s theorem we need to assume that both \( f_{xy} \) and \( f_{yx} \) are continuous. However, we can see from the graphs that they are not.

Notice that the mixed second partial derivatives agree away from \((0, 0)\) where they are continuous. This is good, because Clairaut’s theorem should apply here.

#14.4 45:
By definition, a function is continuous at \((a, b)\) if
\[
\lim_{(x,y) \to (a,b)} f(x, y) = f(a, b).
\]
If \( f \) is differentiable at \((a, b)\) then there is a linear map \( L \) so that \( L(a, b) = f(a, b) \) and
\[
\lim_{(x,y) \to (a,b)} \frac{f(x, y) - L(x, y)}{\sqrt{(x - a)^2 + (y - b)^2}} = 0.
\]
But the fact that \( \sqrt{(x - a)^2 + (y - b)^2} \to 0 \) as \( (x, y) \to (a, b) \) implies that \( f(x, y) - L(x, y) \to 0 \), so
\[
\lim_{(x,y) \to (a,b)} f(x, y) = \lim_{(x,y) \to (a,b)} L(x, y) = f(a, b)
\]
since \( L \) is continuous.

#14.4 46:

a) We use the limit definition of the derivative, so
\[
f_x(0, 0) = \lim_{h \to 0} \frac{h}{h^2 + 0^2} - \frac{0}{h} = 0
\]
and similarly
\[ f_y(0, 0) = \lim_{h \to 0} \frac{0^2 - 0}{h} = 0. \]

However, if we consider the limit of the function as we approach \((0, 0)\) along the path \((t, t)\) we have
\[ \lim_{t \to 0} \frac{t^2}{2t^2} = \frac{1}{2} \neq 0, \]
so \(f\) is not continuous at \((0, 0)\). By question 45 we see that \(f\) cannot be differentiable.

b) The partial derivatives are not continuous because the function itself is not even continuous.