Structure of branch sets of harmonic functions and minimal submanifolds

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A smooth submanifold M is *stationary* if for every family $\{\phi_t\}_{t\in(-1,1)}$ of diffeomorphisms generated by $X \in C_c^\infty(U;\mathbb{R}^{n+m})$ with $K = \operatorname{spt} X$,

$$\frac{d}{dt}\operatorname{Area}(\phi_t(M)\cap K)\Big|_{t=0} = \int_M \operatorname{div}_M X = 0.$$

More generally consider stationary integral varifolds.

$\operatorname{sing} M$ is the set of points where M is not locally a union of smooth immersions.

M is *stable* on $M\setminus\operatorname{sing} M$ if whenever $\operatorname{spt} X \subset U\setminus\operatorname{sing} M$ and X is normal to M,

$$\frac{d^2}{dt^2}\operatorname{Area}(\phi_t(M)\cap K)\Big|_{t=0} \geq 0.$$

In codim $= 1$, letting $X = \zeta\cdot$ unit normal to M,

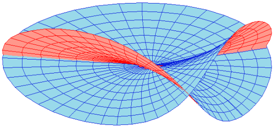
$$\int_M |\nabla^M\zeta|^2 \leq \int_M |A_M|^2\zeta^2 \quad \zeta\in C_c^1(M\setminus\operatorname{sing} M).$$

A *branch point* of a stationary integral n-varifold is a singular point where at least one tangent cone is an n-D plane with integer multiplicity ≥ 2.

Example in codim ≥ 2: $V = \{(z,w) \in \mathbb{C}^2 : w^2 = z^3\}$

Codim $= 1$: Area minimizers do not carry branch points (De Giorgi). However, stable hypersurfaces can carry branch points.

Example: Bour's surface
$\left\{ \left(x_1 - \frac{1}{2}(x_1^2 - x_2^2), -x_2 - x_1 x_2, \frac{4}{3}\,\mathrm{Re}(x_1 + i x_2)^{3/2} \right) : x_1, x_2 \in \mathbb{R} \right\}$



More elaborate constructions of branched q-valued minimal graphs in arbitrary dimension (Simon-Wickramasekera 2007, Krummel 2011).

Little is known about branch points in general.

(For instance, it is possible that $\dim \operatorname{sing} M = n$.)

However, we now have a detailed description (size, local structure,...) in the simplest case of multiplicity two tangent planes.

Theorem (Wickramasekera (2008))

$\exists \epsilon = \epsilon(n, \delta) \in (0, 1)$ *for any orientable immersed stable minimal hypersurface* $M \subset B_2^{n+1}(0)$ *with* $0 \in \overline{M}$ *s.t.*

$$\mathcal{H}^{n-2}(\operatorname{sing} M) < \infty, \quad \frac{\mathcal{H}^n(M)}{\omega_n 2^n} \leq 3 - \delta, \quad \int_{M \cap B_1(0) \times \mathbb{R}} |x^{n+1}|^2 \leq \epsilon.$$

The component of $\overline{M} \cap B_1^{n+1}(0) \times \mathbb{R}$ *containing* 0 *is the graph of a* $C^{1,\mu}$ *single-valued or 2-valued function in* $B_{1/2}^{n+1}(0)$, *where* $\mu = \mu(n, \delta) \in (0, 1)$.

(Wickramasekera (unpublished): $\mathcal{H}^{n-2}(\operatorname{sing} M) < \infty$ *not needed.)*

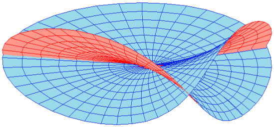
Theorem (Simon-Wickramasekera (2011))

If V is a stationary integral n-varifold with arbitrary codimension and V is the graph of a $C^{1,\mu}$ two-valued function, then the branch set has Hausdorff dimension at most $n-2$.

Theorem (K-Wickramasekera)

If V is a stationary integral n-varifold with arbitrary codimension and V is the graph of a $C^{1,\mu}$ two-valued function, then the branch set is countably $(n-2)$-rectifiable, i.e. is contained in the countable union of a set of $(n-2)$-D Hausdorff measure zero and images of Lipschitz maps $F_j : \mathbb{R}^{n-2} \to \mathbb{R}^n$. (on going)

A *q-valued function* u on a subset $\Omega \subseteq \mathbb{R}^n$ takes an unordered q-tuple $u(X) = \{u_1(X), u_2(X), ..., u_q(X)\}$ at each $X \in \Omega$.



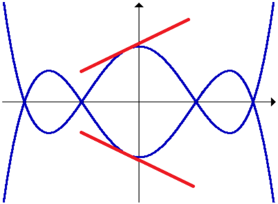
We can not add/multiply general multivalued functions.
Example: Is $\{-1, 1\} + \{3, -3\}$ equal to $\{2, -2\}$ or $\{-4, 4\}$?

Singular set $\mathcal{B}_u$: set of all points $X_0 \in \Omega$ s.t. there is no ball $B_R(X_0)$ s.t. $u = \{u_1, \ldots, u_q\}$ on $B_R(X_0)$ for C^1 single-valued functions $u_1, \ldots, u_q$.

Have metric $\mathcal{G}$ on unordered q-tuples

$$\mathcal{G}(u, v) = \inf_{\sigma \text{ permutation}} \left(\sum_{j=1}^{q} |u_j - v_{\sigma(j)}|^2 \right)^{1/2}$$

Can use this to define continuity, derivatives by affine approximation, Hölder continuity, and limits.



For simplicity, we will first consider as a model case:

$C^{1,\mu}$ **two-valued harmonic functions:** $u \in C^{1,\mu}(\Omega)$ for $\mu \in (0,1]$ such that for every ball $B_R(X_0) \subset \Omega \setminus \mathcal{B}_u$, $u(X) = \{u_1(X), u_2(X)\}$ on $B_R(X_0)$ for single-valued harmonic functions u_1 and u_2.

Arise as approximations of branched solutions to minimal surface system.

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Arise as approximations of branched solutions to minimal surface system.

By replacing $u(X) = \{u_1(X), u_2(X)\}$ with $\{\pm(u_1(X) - u_2(X))/2\}$, may suppose u is symmetric, i.e. $u(X) = \{u_1(X), -u_1(X)\}$ at each $X \in \Omega$.

Will look at

$$\mathcal{K}_u = \{X \in B_1(0) : u(X) = \{0,0\}, Du(X) = \{0,0\}\}.$$

Observe $\mathcal{B}_u \subseteq \mathcal{K}_u$.

Theorem (Simon-Wickramasekera (2011))

If $u \in C^{1,\mu}(\Omega)$ is a two-valued function that is harmonic on $\Omega \setminus \mathcal{B}_u$, then $\dim \mathcal{K}_u \leq n-2$.

Theorem (K-Wickramasekera (preprint))

If $u \in C^{1,\mu}(\Omega)$ is a two-valued function that is harmonic on $\Omega \setminus \mathcal{B}_u$, then $\mathcal{K}_u$ is countably $(n-2)$-rectifiable, i.e. is contained in the countable union of a set of $(n-2)$-D Hausdorff measure zero and images of Lipschitz maps $F_j : \mathbb{R}^{n-2} \to \mathbb{R}^n$.

To each two-valued harmonic function $u \in C^{1,\mu}(B_1(0))$ and $Y \in \mathcal{K}_u$, define frequency function

$$N_{u,Y}(\rho) = \frac{D_{u,Y}(\rho)}{H_{u,Y}(\rho)} = \frac{\rho^{2-n} \int_{B_\rho(Y)} |Du|^2}{\rho^{1-n} \int_{\partial B_\rho(Y)} |u|^2}.$$

$N_{u,Y}(\rho)$ is monotone nondecreasing because

$$\frac{dN_{u,Y}}{d\rho}(\rho) = \frac{2\rho^{1-n}}{H_{u,Y}(\rho)} \int_{\partial B_\rho(Y)} |RD_R u - N_{u,Y}(R) u|^2 \geq 0$$

for $R = |X - Y|$.

$N_{u,Y}(\rho) \equiv \alpha$ constant if and only if u is homogeneous degree α.

May define the frequency $\mathcal{N}_u(Y) = \lim_{\rho \downarrow 0} N_{u,Y}(\rho)$.

Define blow-ups

$$\varphi = \lim_{j\to\infty} \frac{u(Y+\rho_j X)}{\|\mathcal{G}(u(Y+\rho_j X),\{0,0\})\|_{L^2(B_1(0))}}$$

in C^1 and $W^{1,2}$ topologies for appropriate $\rho_j \to 0^+$.

$\varphi \in C^{1,1/2}(B_1(0))$ is a nonzero, homogeneous degree $\mathcal{N}_u(Y)$, symmetric, harmonic two-valued function.

Note that φ may not be unique independent of the sequence ρ_j.

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Simon and Wickramasekera showed by a dimension reduction argument using blow-ups that $\dim \mathcal{K}_u \leq n - 2$.

φ is translation invariant along $S(\varphi) = \{X \in \mathcal{K}_\varphi : \mathcal{N}_\varphi(X) = \mathcal{N}_\varphi(0)\}$.

Obtain stratification

$$\mathcal{S}_0 \subseteq \mathcal{S}_1 \subseteq \mathcal{S}_2 \subseteq \cdots \mathcal{S}_{n-3} \subseteq \mathcal{S}_{n-2} = \mathcal{K}_u, \quad \dim \mathcal{S}_j \leq j,$$

where $\mathcal{S}_j$ is the set $Y \in \mathcal{K}_u$ at which every blow-up φ has $\dim S(\varphi) \leq j$.

Will use a method due to Simon (1993), which was originally applied to multiplicity one classes of minimal submanifolds.

Consider cylindrical two-valued harmonic functions φ of the form

$$\varphi(X_1, X_2, X_3, \ldots, X_n) = \{\pm \operatorname{Re} c(X_1 + iX_2)^\alpha\}$$

after a rotation of $\mathbb{R}^n$ for $c \in \mathbb{C}^m$ and $\alpha = k/2$, $k \geq 3$ an integer.

- Fix a cylindrical two-valued function $\varphi^{(0)} = \varphi^{(0)}(X_1, X_2)$.
- Use coordinates $X = (x, y)$ where $x = (X_1, X_2)$ and $y = (X_3, \ldots, X_n)$.
- Let $u \in C^{1,\mu}(B_1(0))$ be a harmonic two-valued function.
- Let $\varphi \in C^{1,\mu}(B_1(0))$ be cylindrical two-valued function near $\varphi^{(0)}$ in L^2.
- Assume for $\varepsilon \in (0,1)$ to be determined the excess

$$E = \left(\int_{B_1(0)} \mathcal{G}(u(X), \varphi(X))^2 dX \right)^{1/2} < \varepsilon.$$

Want to show for $\vartheta \in (0,1)$ fixed and $\delta = \delta(\vartheta) \in (0,1/8)$ that one of the two alternatives hold: either

1. there is a small δ-gap, i.e. for some $y_0 \in B^{n-2}_{1/2}(0) \subseteq \mathbb{R}^{n-2}$

$$B_\delta(0,y_0) \cap \{X \in \mathcal{K}_u : \mathcal{N}_u(X) \geq \alpha\} = \emptyset, \text{ or}$$

2. for some cylindrical two-valued function $\widetilde{\varphi}$ near $\varphi^{(0)}$ in L^2,

$$\vartheta^{-2\alpha-n} \int_{B_\vartheta(0)} \mathcal{G}(u(X),\widetilde{\varphi}(X))^2 dX \leq C\vartheta^{2\mu} \int_{B_1(0)} \mathcal{G}(u(X),\varphi(X))^2 dX$$

for some constants $C > 0$, $\mu \in (0,1)$.

Consider sequence of u_j and φ_j (in place of u and φ) converging to $\varphi^{(0)}$ in L^2 with no δ_j-gaps for $\delta_j \downarrow 0$ (so Alternative 1 fails) and

$$E_j = \left(\int_{B_1(0)} \mathcal{G}(u_j, \varphi_j)^2 \right)^{1/2} \to 0.$$

On $B_1(0) \cap \{|x| > \tau_j\}$, for $\tau_j \downarrow 0$, we can write

$$v_j = \frac{u_j - \varphi_j}{E_j},$$

regarded as a function on graph of $\varphi^{(0)}$ and show:

- By elliptic estimates, $v_j \to v$ smoothly away from $\mathcal{K}_{\varphi^{(0)}} = \{0\} \times \mathbb{R}^{n-2}$.
- How to control convergence of v_j near $\mathcal{K}_{\varphi^{(0)}} = \{0\} \times \mathbb{R}^{n-2}$.
- v satisfies a decay estimate (similar to one in Alternative 2).

By direct computation using the fact that u is harmonic, we obtain the following new identity:

Lemma

Let $u \in C^{1,\mu}(B_1(0))$ two-valued harmonic, $Y \in B_1(0)$, and $\alpha \in \mathbb{R}$. Then

$$\frac{d}{d\rho}(\rho^{-2\alpha}(D_{u,Y}(\rho) - \alpha H_{u,Y}(\rho))) = 2\rho^{-n} \int_{\partial B_\rho(Y)} |RD_R u - \alpha u|^2,$$

where $R = |X - Y|$ and

$$D_{u,Y}(\rho) = \rho^{2-n} \int_{B_\rho(Y)} |Du|^2, \quad H_{u,Y}(\rho) = \rho^{1-n} \int_{\partial B_\rho(Y)} |u|^2.$$

Using the lemma on the previous slide, we prove our main estimate, from which the other L^2 estimates follow.

Lemma (Main Estimate)

Given $\gamma \in (0,1)$, if u and $\varphi = \varphi(X_1, X_2)$ are sufficiently close to $\varphi^{(0)}$ in L^2 and $0 \in \mathcal{K}_u$ with $\mathcal{N}_u(0) \geq \alpha$, then

$$\int_{B_\gamma(0)} R^{2-n} \left| \frac{\partial(u/R^\alpha)}{\partial R} \right|^2 \leq C \int_{B_1(0)} \mathcal{G}(u,\varphi)^2$$

for $C = C(n, \varphi^{(0)}, \alpha, \gamma) > 0$.

Theorem

Given $\gamma, \tau, \sigma \in (0,1)$, if u and $\varphi = \varphi(X_1, X_2)$ are sufficiently close to $\varphi^{(0)}$ in L^2 and $Z = (\xi, \eta) \in \mathcal{K}_u \cap B_{1/2}(0)$ with $\mathcal{N}_u(Z) \geq \alpha$, then

$$|\xi|^2 + \int_{B_\gamma(0)} |D_y u|^2 + \int_{B_\gamma(0)} R_Z^{2-n} \left| \frac{\partial (u/R_Z^\alpha)}{\partial R_Z} \right|^2 \leq C \int_{B_1(0)} \mathcal{G}(u,\varphi)^2$$

for $C = C(n, \varphi^{(0)}, \alpha, \gamma) > 0$, where $R_Z = |X - Z|$, and

$$\int_{B_\gamma(0)} \frac{\mathcal{G}(u,\varphi)^2}{|X-Z|^{n-1-\sigma}} + \int_{B_\gamma(0) \cap \{|x| > \tau\}} \frac{|u(X) - \varphi(X) - D_x \varphi(X) \cdot \xi|^2}{|X-Z|^{n+2\alpha-\sigma}} \leq C \int_{B_1(0)} \mathcal{G}(u,\varphi)^2$$

for $C = C(n, \varphi^{(0)}, \alpha, \gamma, \sigma) > 0$.

One of the L^2 estimates easily implies when $\delta > 0$, u, φ are close to $\varphi^{(0)}$ in L^2, and there are no δ-gaps,

$$\int_{B_{1/4}(0)} \frac{\mathcal{G}(u,\varphi)^2}{r_\delta^{1-\sigma}} \leq C \int_{B_1(0)} \mathcal{G}(u,\varphi)^2,$$

where $r_\delta = \max\{|x|, \delta\}$ and $C = C(n, \varphi^{(0)}, \alpha, \gamma, \sigma) > 0$.

Hence u does not concentrate near $\{0\} \times \mathbb{R}^{n-2}$: if τ is small, $\tau > \delta$,

$$\int_{B_{1/4}(0) \cap \{|x| < \tau\}} \mathcal{G}(u,\varphi)^2 \leq C\tau^{1-\sigma} \int_{B_1(0)} \mathcal{G}(u,\varphi)^2.$$

Follows that $v_j \to v$ in $L^2(B_{1/4}(0))$.

By dividing by E_j^2 in the L^2 estimates for u_j and φ_j and letting $j \to \infty$, we obtain similar estimates for the blow-ups v in terms of $\int_{B_1(0)} |v|^2$.

Let ψ_ρ be the $L^2(B_\rho(0))$ projection of v onto homogeneous degree α functions.

Here $L^2(B_\rho(0))$ makes sense since v is a function on $\operatorname{graph} \varphi^{(0)}$.

Using the estimates on v we can characterize ψ_ρ and show that

$$\int_{B_{\rho/4}(0)} R^{2-n} \left| \frac{\partial(v/R^\alpha)}{\partial R} \right|^2 \leq C\rho^{-2\alpha-n} \int_{B_\rho(0)} |v - \psi_\rho|^2,$$

$$\rho^{-2\alpha-n} \int_{B_\rho(0)} |v - \psi_\rho|^2 \leq C \int_{B_\rho(0)\setminus B_{\rho/4}(0)} R^{2-n} \left| \frac{\partial(v/R^\alpha)}{\partial R} \right|^2.$$

The decay estimate for v will follow.

Recall that

$$\int_{B_{\rho/4}(0)} R^{2-n} \left|\frac{\partial(v/R^\alpha)}{\partial R}\right|^2 \leq C\rho^{-2\alpha-n} \int_{B_\rho(0)} |v-\psi_\rho|^2,$$

$$\rho^{-2\alpha-n} \int_{B_\rho(0)} |v-\psi_\rho|^2 \leq C \int_{B_\rho(0)\setminus B_{\rho/4}(0)} R^{2-n} \left|\frac{\partial(v/R^\alpha)}{\partial R}\right|^2,$$

$\psi_\rho = L^2(B_\rho(0))$ projection of v onto homo. deg. α functions.

Combining and using "hole-filling" (add integral over $B_{\rho/4}$ to fill in $B_\rho \setminus B_{\rho/4}$)

$$\int_{B_{\rho/4}(0)} R^{2-n} \left|\frac{\partial(v/R^\alpha)}{\partial R}\right|^2 \leq \gamma \int_{B_\rho(0)} R^{2-n} \left|\frac{\partial(v/R^\alpha)}{\partial R}\right|^2.$$

for some $\gamma \in (0,1)$. Iterate to get for some $\mu \in (0,1)$,

$$\vartheta^{-n-2\alpha} \int_{B_\vartheta(0)} |v-\psi_\vartheta|^2 \leq C\vartheta^{2\mu} \int_{B_1(0)} |v-\psi_1|^2.$$

We showed for $\vartheta \in (0,1)$ and $\delta = \delta(\vartheta) \in (0,1/8)$ that either

1. there is a small gap, i.e. for some $y_0 \in \mathbb{R}^{n-2}$

$$B_\delta(0,y_0) \cap \{X \in \mathcal{K}_u : \mathcal{N}_u(X) \geq \alpha\} = \emptyset, \text{ or}$$

2. for some cylindrical homogeneous two-valued function $\widetilde{\varphi}$ near $\varphi^{(0)}$,

$$\vartheta^{-2\alpha-n} \int_{B_\vartheta(0)} \mathcal{G}(u(X),\widetilde{\varphi}(X))^2 dX \leq C\vartheta^{2\mu} \int_{B_1(0)} \mathcal{G}(u(X),\varphi(X))^2 dX$$

   for some constants $C > 0$, $\mu \in (0,1)$.

If no small gap, would get $\mathcal{K}_u$ is a $C^{1,\mu}$ $(n-2)$-D submanifold.

In general we get $(n-2)$-rectifiability of $\mathcal{K}_u$ together with locally finiteness properties of measure.

Special cases where no small gaps occur:

- Points locally $\mathcal{N}_u \equiv 1/2 + k$ constant for $k \in \mathbb{Z}_+$.
- Points where $\mathcal{N}_u = 3/2$.

Theorem

If $u \in C^{1,\mu}(B_1(0))$ is a nonzero, symmetric two-valued function such that u is harmonic on $B_1(0) \setminus \mathcal{B}_u$ and $\mathcal{N}_u(Y) \equiv 1/2 + k$ for all $Y \in \mathcal{B}_u$ and some constant integer $k \geq 1$, then not only is $\mathcal{B}_u$ a $C^{1,\tau}$ submanifold but $\mathcal{B}_u$ is real analytic. (on going)

Idea of proof: Reduce to $\mathcal{N}_u \equiv 3/2$. Straighten out $\mathcal{B}_u$ via a Legendre-type transformation and inductively apply a Schauder estimate for transformed functions. Need rate of convergence to blow-ups for this.

Future work: Show we cannot have $\mathcal{B}_u \neq \emptyset$ and locally $\mathcal{N}_u \equiv k$ on $\mathcal{B}_u$ constant for $k \in \mathbb{Z}_+$.

Dirichlet minimizing two-valued functions: $u \in W^{1,2}(\Omega)$ such that

$$\int_\Omega |Du|^2 \leq \int_\Omega |Dv|^2$$

whenever $v \in W^{1,2}(\Omega)$ two-valued with $\{X : u(X) \neq v(X)\}$ compact and $u = v$ on $\partial\Omega$.

Arise as approximations of area minimizers in Almgren's and De Lellis-Sparado's proof that $\dim \mathrm{sing}\, T \leq n - 2$ for an area minimizing n-current T.

$W^{1,2}$ Dirichlet minimizing, $C^{1,\mu}$ harmonic are not equivalent:

- $u(z) = \{\pm \operatorname{Re} z^{3/2}\}$ is $C^{1,1/2}$ harmonic but not minimizing.
- $u(z) = \{\pm z^{1/2}\}$ is minimizing but not C^1.

(Here we use $\mathbb{C} \cong \mathbb{R}^2$.)

Theorem (K-Wickramasekera)

If $u \in W^{1,2}(\Omega)$ is Dirichlet-minimizing, then $\mathcal{B}_u$ is countably $(n-2)$-rectifiable.

Proof uses the same blow-up method.

As before we assume u is nonzero and symmetric, i.e. $u(X) = \{\pm u_1(X)\}$. We replace $\mathcal{K}_u$ with $\Sigma_u = \{X : u(X) = \{0,0\}\}$.

Now have additionally $\alpha = 1/2, 1$. To characterize homogeneous projections of blow-ups v for $\alpha = 1/2$ need to show that

$$\lim_{r \downarrow 0} \frac{\partial^2}{\partial r \partial y} \int_{S^1} rv(re^{i\theta}, y, \varphi^{(0)}(re^{i\theta}, y)) D_j \varphi^{(0)}(re^{i\theta}, y) d\theta = 0$$

for $j = 1, 2$.

Recall that we wanted to prove:

Theorem (K-Wickramasekera)

If V is a stationary integral n-varifold with arbitrary codimension and V is the graph of a $C^{1,\mu}$ two-valued function, then the branch set is countably $(n-2)$-rectifiable.

At a branch point Y, write V as the graph of $\tilde{u}(X) = \{\tilde{u}_1(X), \tilde{u}_2(X)\}$ over the tangent plane of V at Y. Compute frequency and blow-ups φ at Y relative to $\tilde{u}_s(X) = \{\pm(\tilde{u}_1(X) - \tilde{u}_2(X))/2\}$.

Need to control the rotation of tangent planes so that we can compute writing M as the graph of u over a *fixed* plane.

Also need to control that the minimal surface system is quasilinear, so $\Delta u_s = f$ for some function f and $u_s/\Lambda \approx \varphi$ for scaling factor Λ.

Therefore we use

$$\text{Excess} = \int_{B_1(0)} \mathcal{G}(u_s/\Lambda, \varphi)^2 + \Lambda^{-2} \int_{B_{7/8}(0)} R^{4-n-2\alpha+2\varepsilon_1} |f|^2 + \Lambda^{-2} \sup_{Y \in \mathcal{K}_u} |Du(Y)| \int_{B_{7/8}(0)} |u_s|^2$$

Use modified blow-up method to prove branch set of V is countably $(n-2)$-rectifiable.

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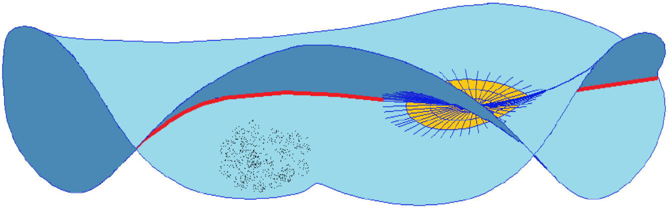
Use modified blow-up method to prove branch set of V is countably $(n-2)$-rectifiable.

Let V be a stationary, stable integral n-varifold in $B_1(0) \subset \mathbb{R}^{n+1}$ associated with a current T with $\partial T = 0$. $\{p \in \operatorname{spt} V : \Theta(\|V\|, p) < 3\}$ is a relatively open set in $\operatorname{spt} V$ that stratifies as follows:



1. Regular set: Points where $\operatorname{spt} V$ is smooth embedded.
2. Self-intersections: $(n-1)$-D C^1 submanifold. Tangent cones of V are pairs of transverse planes.
3. Top-dim part of singular set: Countably $(n-2)$-rectifiable set. V locally like the graph of $\operatorname{Re} cz^\alpha$, $\alpha \in \{3/2, 2, 5/2, 3, 7/2, \ldots\}$.
4. **Low-dim part of singular set** (black spots): Dimension $\leq (n-3)$.

Future work will consider Dirichlet energy minimizing q-valued function and $C^{1,\mu}$ harmonic q-valued functions, $q \geq 3$.

For $C^{1,\mu}$ harmonic q-valued functions: We can prove a $C^{1,1/q}$ regularity theory and countably $(n-2)$-rectifiability of the branch set via an inductive argument on q.

For this we need to extend the Liousville-type result and Schauder estimate of Simon-Wickramasekera from $q = 2$; for this we assume the set of branch points Y where $u(Y) \neq q[\![0]\!]$ is rectifiable and thus has zero 2-capacity.