Lecture 3: Multivalued functions - Part II

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Lebesgue functions. For $1 \leq p \leq \infty$ and $\Omega \subseteq \mathbb{R}^n$ Lebesgue measurable, we define $L^p(\Omega, \mathcal{A}_q(\mathbb{R}^n))$ to be the space of all Lebesgue measurable $u : \Omega \to \mathcal{A}_q(\mathbb{R}^n)$ such that

$$\|u\|_{L^p(\Omega)} = \|\mathcal{G}(u, q[\![0]\!])\|_{L^p(\Omega)} = \left(\int_\Omega |u|^p \right)^{1/p} < \infty$$

when $1 \leq p < \infty$ and

$$\|u\|_{L^\infty(\Omega)} = \|\mathcal{G}(u, q[\![0]\!])\|_{L^\infty(\Omega)} = \sup_\Omega |u| < \infty$$

when $p = \infty$.

Continuous multivalued functions. Again notice that since $\mathcal{A}_q(\mathbb{R}^n)$ is a metric space, it is clear what we mean by a q-valued function u is continuous, Hölder continuous, or Lipschitz. In particular, $u : \Omega \to \mathcal{A}_q(\mathbb{R}^n)$ is continuous if at each $y \in \Omega$

$$\lim_{x \to y} \mathcal{G}(u(x), u(y)) = 0.$$

We let $C^0(\Omega, \mathcal{A}_q(\mathbb{R}^n))$ denote the set of continuous q-valued functions $u : \Omega \to \mathcal{A}_q(\mathbb{R}^n)$. We equip $C^0(\Omega, \mathcal{A}_q(\mathbb{R}^n))$ with the metric

$$\operatorname{dist}_{C^0}(u, v) = \sup_{x \in \Omega} \mathcal{G}(u(x), v(x))$$

for $u, v : \Omega \to \mathcal{A}_q(\mathbb{R}^n)$. Thus for $u_k, u : \Omega \to \mathcal{A}_q(\mathbb{R}^n)$, we say that $u_k \to u$ in $C^0(\Omega, \mathcal{A}_q(\mathbb{R}^n))$ if

$$\lim_{k \to \infty} \sup_{x \in \Omega} \mathcal{G}(u_k(x), u(x)) = 0.$$

Moreover, it is clear what is meant by Hölder and Lipschitz q-valued functions. For $\alpha \in (0, 1]$, we say the q-valued function $u : \Omega \to \mathcal{A}_q(\mathbb{R}^n)$ is Hölder continuous with exponent α if

$$[u]_{\alpha,\Omega} = \sup_{x,y \in \Omega,\, x \neq y} \frac{\mathcal{G}(u(x), u(y))}{|x - y|^\alpha} < \infty$$

and we let $C^{0,\alpha}(\Omega, \mathcal{A}_q(\mathbb{R}^n))$ denote the set of all such q-valued functions u. We say u is Lipschitz when $\alpha = 1$ and we let $\operatorname{Lip} u = [u]_{1,\Omega}$. We have the following special case of Arzela-Ascoli:

Theorem 1 (Arzela-Ascoli)**.** *Let Ω be a compact subset of $\mathbb{R}^m$. If $u_k : \Omega \to \mathcal{A}_q(\mathbb{R}^n)$ is a sequence of q-valued functions with*

$$\sup_k \left(\sup_\Omega |u_k| + [u_k]_{\alpha,\Omega} \right) < \infty$$

then there exists a subsequence $u_{k'}$ *and* q*-valued function* $u : \Omega \to \mathcal{A}_q(\mathbb{R}^n)$ *such that* $u_{k'} \to u$ *in* $C^0(\Omega, \mathcal{A}_q(\mathbb{R}^n))$ *and*

$$[u]_{\alpha,\Omega} \leq \liminf_{k\to\infty} [u_k]_{\alpha,\Omega}.$$

We also have the following Lipschitz extension lemma:

Lemma 2 (Lipschitz extension)**.** *Let* $\Omega \subseteq \mathbb{R}^m$ *and* $u : \Omega \to \mathcal{A}_q(\mathbb{R}^n)$ *be any Lipschitz* q*-valued function. There exists an extension* $\overline{u} : \mathbb{R}^n \to \mathcal{A}_q(\mathbb{R}^n)$ *such that*

$$\overline{u}|_\Omega = u, \quad \operatorname{Lip} \overline{u} \leq C(m,q) \operatorname{Lip} u.$$

When u *is bounded, we also have*

$$\sup_{x\in\Omega} \mathcal{G}(\overline{u}(x), a) \leq C(m,q) \sup_{x\in\Omega} \mathcal{G}(u(x), a) \text{ for all } a \in \mathcal{A}_q(\mathbb{R}^n).$$

Sketch of proof. Proceed by induction on q. In the case $q = 1$ and $n = 1$, we can define

$$\overline{u}(x) = \sup_{y\in\Omega} (u(y) - \operatorname{Lip} u\, |x-y|) \quad \text{for } x \in \mathbb{R}^n$$

and $\operatorname{Lip} \overline{u} \leq \operatorname{Lip} u$. It is then very easy to deduce for $n > 1$ the existence of a Lipschitz extension with $\operatorname{Lip} \overline{u} \leq \sqrt{n} \operatorname{Lip} u$. The existence of an extension with the same Lipschitz constant is a classical, but subtle, result of Kirszbraun, see 2.10.43 in Federer.

In the induction step where $q > 1$, decompose $\mathbb{R}^m \setminus \Omega$ into a Whitney decomposition of cubes C_j with the following properties:

(i) each C_j is a closed cube whose side length $\ell_j = 2^{k_j}$ for some $k_j \in \mathbb{Z}$ and whose vertices are integer multiples of 2^k,

(ii) distinct cubes have disjoint interiors, and

(iii) $\frac{1}{c(m)} \operatorname{dist}(C_k, \Omega) \leq \ell_k \leq c(m) \operatorname{dist}(C_k, \Omega)$ for some constant $c(m) \in (0, \infty)$.

For each vertex x of some C_j, let $\overline{u}(x) = u(y)$ where $y \in \Omega$ is the closest point to x. We then proceed to inductively extend $\overline{u}$ to each edge, face,..., k-cell, etc. For each k-cell F, to extend $\overline{u}|_{\partial F}$ to $\overline{u}|_F$, after a Lipschitz change of coordinates we can take F to be a ball. Fix $x_0 \in \partial F$. If values $\overline{u}_i(x_0), \overline{u}_j(x_0)$ are relatively far apart for some i, j in the sense that

$$\mathcal{G}(\overline{u}_i(x_0), \overline{u}_j(x_0)) > 3 \operatorname{Lip} \overline{u} \operatorname{diam}(F),$$

then $\overline{u}$ decomposes into two simpler multivalued functions on ∂F each of which have a Lipschitz extension to F by the induction hypothesis on q. Otherwise, we can extend $\overline{u}$ by

$$\overline{u}(x) = \sum_{i=1}^q [\![|x|\, u_i(x) + (1-|x|)\, u_1(x_0)]\!] \quad \text{for } x \in F.$$

One then at each stage on checks the estimates, e.g. $\operatorname{Lip}(\overline{u}|_F) \leq C(m,q) \operatorname{Lip}(\overline{u}|_{\partial F})$. $\square$

Derivatives. A q-valued function $L : \mathbb{R}^m \to \mathcal{A}_q(\mathbb{R}^n)$ is affine if L takes the form

$$L(x) = \sum_{i=1}^{q} [\![a_i + B_i x]\!]$$

for all $x \in \mathbb{R}^m$ for some $a_i \in \mathbb{R}^n$ and real-valued $n \times m$ matrices B_i.

Let $\Omega \subseteq \mathbb{R}^m$ and $u : \Omega \to \mathcal{A}_q(\mathbb{R}^n)$ be a q-valued function. We say that u is affine approximable at $x_0 \in \Omega$ if there exists an affine q-valued function

$$L(x) = \sum_{i=1}^{q} [\![u_i(x_0) + B_i x]\!]$$

such that

$$\lim_{x \to x_0} \frac{\mathcal{G}(u(x), u(x_0))}{|x - x_0|} = 0.$$

We say that u is strongly affine approximable at $x_0 \in \Omega$ if additionally $B_i = B_j$ whenever $u_i(x_0) = u_j(x_0)$. The concept of strongly affine approximable will help simplify proofs (see for instance Rademacher's theorem below). We call L the affine approximation of u at x_0. Shall denote the derivative $Du(x_0)$ and affine approximation L of u by

$$Du(x) = \sum_{i=1}^{q} [\![Du_i(x_0)]\!], \quad L(x) = \sum_{i=1}^{q} [\![u_i(x_0) + Du_i(x_0) \cdot (x - x_0)]\!],$$

for $m \times n$ matrices $Du_i(x_0)$ with the convention that $Du_i(x_0)$ is paired with $u_i(x_0)$.

As an example, $u : \mathbb{R} \to \mathcal{A}_q(\mathbb{R})$ given by

$$u(x) = [\![x]\!] + [\![-x]\!]$$

for $x \in \mathbb{R}$ is affine approximable at the origin and is its own affine approximation but u is not strongly affine approximable at the origin.

Theorem 3 (Rademacher)**.** *Let $u : \Omega \to \mathcal{A}_q(\mathbb{R}^n)$ be a Lipschitz q-valued function. Then u is strongly affine approximable almost everywhere.*

Sketch of proof. Argue by induction on q. For the induction step, let

$$\Omega_0 = \{x \in \Omega : u(x) = q[\![u_1(x)]\!] \text{ for some value } u_1(x) \in \mathbb{R}^m\}.$$

On sufficiently small relatively open neighborhoods $U \subseteq \Omega \setminus \Omega_0$, $u = [\![u_K]\!] + [\![u_L]\!]$ for two simpler Lipschitz multivalued functions u_K, u_L and by the induction hypothesis u_K, u_L are strongly affine approximable a.e. Let $u_1 : \Omega_0 \to \mathbb{R}^n$ be the function such that $u(x) = q[\![u_1(x)]\!]$ for all $x \in \Omega_0$. Then u_1 is Lipschitz and extends to a Lipschitz function on all of $\mathbb{R}^m$ and so u_1 is differentiable a.e. One can show that u is strongly affine approximable at points where Ω_0 has density one and u_1 is differentiable. $\square$

Warning! One cannot in general add, subtract, or multiply q-valued functions!
Since there is no canonical ordering of the values of q-values, it is not clear how to add, subtract, or multiply a pair of them in order to obtain a q-valued result. For instance, should

$$\Big([\![(1,0)]\!] + [\![(0,1)]\!]\Big) + \Big([\![(-1,0)]\!] + [\![(0,-1)]\!]\Big) = [\![(0,0)]\!] + [\![(0,0)]\!] \quad \text{or} \quad [\![(1,-1)]\!] + [\![(-1,1)]\!]\,?$$

More importantly, multivalued harmonic functions have branching behavior. Consider $u(x) = \pm\operatorname{Re}(x_1 + ix_2)^{3/2}$ and $v(x) = \pm\operatorname{Re}(x_1 + ix_2 - 1)^{3/2}$ as two-valued harmonic functions. (These functions are not Dirichlet minimizing but are $C^{1,1/2}$ and thus are important to understanding stable minimal hypersurfaces. We could also consider $u(x) = \pm(x_1 + ix_2)^{3/2}$ and $v(x) = \pm(x_1 + ix_2 - 1)^{3/2}$, which are $C^{1,1/2}$ and Dirichlet minimizing.) What should $u + v$ be? Well, if we want $u+v$ to be C^1 harmonic, we have precisely two options far away from the origin, say on $\mathbb{R}^2 \setminus B_2(0)$. If we extend these continuous sums $u + v$ to a neighborhood of the origin, then there is clearly some ambiguity and whatever sum $u + v$ we decide on will not be C^1 or harmonic. Thus in adding u and v we lost regularity/structure.

Now I am fibbing a bit. If a pair of q-valued harmonic functions u and v are close together, one could use elliptic estimates to define $u - v$. I do this with Neshan in our joint work. But one needs to be very careful and precise in doing this! Also, in the case that codimension $m = 1$, there is a canonical ordering for a q-value $a = \sum_{i=1}^{q} [\![a_i]\!]$ in $\mathcal{A}_q(\mathbb{R})$ given by putting the values a_i in nondecreasing order $a_1 \leq a_2 \leq \cdots \leq a_q$. This ordering behaves nicely relative to the metric $\mathcal{G}$ and induces sums, products, and selections of multivalued functions. However, this fails for $m > 1$. And as the example $u(x) = \pm(x_1 + ix_2)^{3/2}$ above indicates, if u is C^1 and/or harmonic and $u(x) = \sum_{i=1}^{q} [\![u_i(x)]\!]$ with u_i in increasing order, then the graphs of the functions u_i can have "corners" and cease to be C^1 and/or harmonic. Hence one tends not to want to use this sort of ordering when working with multivalued harmonic functions.