Lecture 2: Multivalued functions - Part I

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Multivalues. Let $q, n \geq 1$ be integers. Informally, a q-value a consists of an unordered collection of values $a_1, a_2, \ldots, a_q \in \mathbb{R}^n$, possibly repeating. We can represent this as a 0-dimensional current

$$a = \sum_{i=1}^{q} [\![a_i]\!]$$

where $[\![a_i]\!]$ denotes the Dirac point mass at a_i. For those familiar with geometric measure theory, q-values can be regarded as 0-dimensional integral currents. For those not familiar with geometric measure theory, we can just take this as notation. The sum here can be regarded as a "union" operator that takes the union of collections of points and adds their multiplicities. For example,

$$a = [\![7]\!] + [\![7]\!] + [\![10]\!] = 2[\![7]\!] + [\![10]\!]$$

as a union of the points $7, 7, 10$ to form a 3-value a with multiplicity two at 7 and one at 10. This is useful for keeping track of multiplicities.

We let $\mathcal{A}_q(\mathbb{R}^n)$ denote the space of all q-values $a = \sum_{i=1}^q [\![a_i]\!]$ with $a_i \in \mathbb{R}^n$. Notice that when $q = 1$, we can identify $\mathcal{A}_1(\mathbb{R}^n) = \mathbb{R}^n$.

We equip $\mathcal{A}_q(\mathbb{R}^n)$ with a complete metric $\mathcal{G}$ given by

$$\mathcal{G}(a, b) = \min_{\text{permutations } \sigma \text{ of } \{1,2,\ldots,q\}} \left(\sum_{i=1}^{q} |a_i - b_{\sigma(i)}|^2 \right)^{1/2}.$$

for q-values $a = \sum_{i=1}^q [\![a_i]\!]$ and $b = \sum_{i=1}^q [\![b_i]\!]$. In words, we take the q-values a and b and order their values. Then we take the total distance between their respective values a_i and b_i. But recall that I said q-values are unordered! So we take the minimum of all the ways to order the values of a and b, giving us our metric. As notation, it is convenient to define

$$|a| = \mathcal{G}(a, q[\![0]\!]) = \left(\sum_{i=1}^{q} |a_i|^2 \right)^{1/2}$$

for every q-values $a = \sum_{i=1}^q [\![a_i]\!]$.

Exercise 1. *Suppose $n = 1$. Let $a = \sum_{i=1}^q [\![a_i]\!]$ and $b = \sum_{i=1}^q [\![b_i]\!]$ be q-values in $\mathcal{A}_q(\mathbb{R})$ and put their values in nondecreasing order $a_1 \leq a_2 \leq \cdots \leq a_q$ and $b_1 \leq b_2 \leq \cdots \leq b_q$. Show that*

$$\mathcal{G}(a, b) = \left(\sum_{i=1}^{q} |a_i - b_i|^2 \right)^{1/2}.$$

Exercise 2. *Given a q-value* $a = \sum_{i=1}^{q} [\![a_i]\!]$, *let*

$$a_{\text{avg}} = \frac{1}{q} \sum_{i=1}^{q} a_i \in \mathbb{R}^n, \quad a_{\text{sym}} = \sum_{i=1}^{q} [\![a_i - a_{\text{avg}}]\!] \in \mathcal{A}_q(\mathbb{R}^n)$$

so that a_{avg} *is the average of the values of* a *and the average of the values of* a_{sym} *is zero. Verify that* $|a|^2 = q\,|a_{\text{avg}}|^2 + |a_{\text{sym}}|^2$.

Multivalued functions. Let $q, m, n \geq 1$ be integers and $\Omega \subseteq \mathbb{R}^m$. A q-valued function is a map $u : \Omega \to \mathcal{A}_q(\mathbb{R}^n)$. We will represent a q-valued function u as

$$u(x) = \sum_{i=1}^{q} [\![u_i(x)]\!]$$

at $x \in \Omega$ unless stated otherwise.

Given a collection of q_i-valued functions $u_i : \Omega \to \mathcal{A}_{q_i}(\mathbb{R}^n)$ for $i = 1, 2, \ldots, N$, I can define a q-valued function $u : \Omega \to \mathcal{A}_q(\mathbb{R}^n)$, where $q = \sum_{i=1}^{N} q_i$, by

$$u(x) = \sum_{i=1}^{N} u_i(x) = \sum_{i=1}^{N} \sum_{j=1}^{q_i} [\![u_{ij}(x)]\!] \quad \text{where} \quad u_i(x) = \sum_{j=1}^{q_i} [\![u_{ij}(x)]\!]$$

for $x \in \Omega$. Recall that the sum $u(x) = \sum_{i=1}^{N} u_i(x)$ is taken as sums of Dirac masses. We are not adding the values of each $u_i(x)$ but rather letting $u(x)$ consist of every point that is a value of some $u_i(x)$ and adding up the multiplicities at that point. To distinguish this "union" of u_i from the addition of values of u_i, we shall as a slight abuse of notation denote the "union" of the u_i by

$$u(x) = \sum_{i=1}^{N} [\![u_i(x)]\!].$$

As an example,

$$u(x) = [\![(x_1 + ix_2)^{1/2}]\!] + 10[\![(x_1 + ix_2)^{1/3}]\!]$$

is a $2 + 10 \cdot 3 = 32$ valued function. When $q_1 = q_2 = \cdots q_N = 1$, we say $u_1, u_2, \ldots, u_q$ is a selection of u on Ω.

Measurable multivalued functions. Notice that since $\mathcal{A}_q(\mathbb{R}^n)$ is a metric space, it is clear what we mean by a q-valued function $u : \Omega \to \mathcal{A}_q(\mathbb{R}^n)$ is Lebesgue measurable. In particular, u is Lebesgue measurable, or measurable, if for every Borel set $B \subseteq \mathcal{A}_q(\mathbb{R}^n)$, $u^{-1}(B)$ is Lebesgue measurable.

The following result is useful in simplifying notation.

Proposition 1. *Let* $u : \Omega \to \mathcal{A}_q(\mathbb{R}^n)$ *be a Lebesgue measurable q-valued function. Then, there exist a selection of Lebesgue measurable functions* $u_i : \Omega \to \mathbb{R}^m$.

Proof. Let us proceed by induction on q. The case $q = 1$ is trivial. Assume that for some integer $q \geq 2$, for every $q' = 1, 2, \ldots, q-1$ and every measurable q'-valued function has a measurable selection. Let $u : \Omega \to \mathcal{A}_q$ be measurable. We want to show that u has a measurable selection on Ω.

Observe that if $\Omega = \bigcup_{i=1}^{\infty} \Omega_i$ for a countable collection of measurable subsets $\Omega_i \subset \mathbb{R}^n$ and each $u|_{\Omega_i}$ has a measurable selection, then u also has a measurable selection.

Let $A_0 = \{q[\![a_1]\!] : a_1 \in \mathbb{R}^m\} \subseteq \mathcal{A}_q(\mathbb{R}^n)$. Let $\Omega_0 = u^{-1}(A_0)$ be the set of points $x \in \Omega$ such that $u(x) = q[\![u_1(x)]\!]$ for some value $u_1(x) \in \mathbb{R}^m$. A_0 is closed, so Ω_0 is measurable. Obviously $u|_{\Omega_0}$ has a selection.

Fix $a \in \mathcal{A}_q \setminus A_0$ and write $a = \sum_{i=1}^{q} [\![a_i]\!]$ for an ordered collection of values a_i. Partition $\{1, 2, \ldots, q\}$ into nonempty sets I_K and I_L with cardinality K and L respectively such that

$$|a_i - a_j| > 0 \text{ for all } i \in I_K,\ j \in I_L.$$

Now for every $b \in \mathcal{A}_q$, there exists a permutation σ such that

$$\mathcal{G}(a, b) = \left(\sum_{i=1}^{q} |a_i - b_{\sigma(i)}|^2 \right)^{1/2}$$

and for a sufficiently small open neighborhood V_a of a in $\mathcal{A}_q$ the maps

$$b \in V_a \mapsto \sum_{i \in I_K} [\![b_{\sigma(i)}]\!], \quad b \in V \mapsto \sum_{i \in I_L} [\![b_{\sigma(i)}]\!]$$

are continuous. Since V_a is open, $u^{-1}(V_a)$ is measurable. By the induction hypothesis, $u|_{V_a}$ has a measurable selection.

Notice that we can cover $\mathcal{A}_q \setminus A_0 = \bigcup_{i=1}^{\infty} V_{a^i}$ for a countable collection of $a^i \in \mathcal{A}_q \setminus A_0$. Since $u|_{A_0}$ and $u|_{V_{a^i}}$ all have measurable selections, so does u. □