1. True or false? Justify your answers.
   (a) If \( z \) is orthogonal to \( u_1 \) and to \( u_2 \) and if \( W = \text{Span}\{u_1, u_2\} \), then \( z \) must be in \( W^\perp \).
   (b) For each \( y \) and each subspace \( W \), the vector \( y - \text{proj}_W y \) is orthogonal to \( W \).
   (c) The orthogonal projection \( \hat{y} \) of \( y \) onto a subspace \( W \) can sometimes depend on the orthogonal basis for \( W \) used to compute \( \hat{y} \).
   (d) If \( y \) is in a subspace \( W \), then the orthogonal projection of \( y \) onto \( W \) is \( y \) itself.
   (e) If the columns of an \( n \times p \) matrix \( U \) are orthonormal, then \( U U^T y \) is the orthogonal projection of \( y \) onto the column space of \( U \).
   (f) If \( b \) is in the column space of \( A \), then every solution of \( A x = b \) is a least-squares solution.
   (g) The least-squares solution of \( A x = b \) is a list of weights that, when applied to the columns of \( A \), produces the orthogonal projection of \( b \) onto \( \text{Col} \ A \).
   (i) The normal equations always provide a reliable method for computing least-squares solutions.
   (j) If \( \hat{x} \) is a least-squares solution of \( A x = b \), then \( \hat{x} = (A^T A)^{-1} A^T b \).

2. Find the best approximation to \( z \) by vectors of the form \( c_1 v_1 + c_2 v_2 \):
   \[
   z = \begin{bmatrix} 3 \\ -7 \\ 2 \\ 3 \end{bmatrix}, \quad v_1 = \begin{bmatrix} 2 \\ -1 \\ -3 \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}.
   \]

3. Suppose \( A \) is \( m \times n \) with linearly independent columns and \( b \) is in \( \mathbb{R}^m \). Use the normal equations to produce a formula for \( \hat{b} \), the projection of \( b \) onto \( \text{Col} \ A \).

4. Find a least-squares solution of
   \[
   \begin{bmatrix} 2 & 1 \\ -2 & 0 \\ 2 & 3 \end{bmatrix} x = \begin{bmatrix} -5 \\ 8 \\ 1 \end{bmatrix}
   \]
   by constructing the normal equations for \( \hat{x} \) and solving for \( \hat{x} \).

5. Describe all least-squares solutions of the equation
   \[
   \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} x = \begin{bmatrix} 1 \\ 3 \\ 8 \\ 2 \end{bmatrix}.
   \]

6. Find the orthogonal projection of \( b \) onto \( \text{Col} \ A \) and the least squares solution of \( A x = b \) for
   \[
   A = \begin{bmatrix} 4 & 0 & 1 \\ 1 & -5 & 1 \\ 6 & 1 & 0 \\ 1 & -1 & -5 \end{bmatrix}, \quad b = \begin{bmatrix} 9 \\ 0 \\ 0 \\ 0 \end{bmatrix}.
   \]

7. Let \( A \) be an \( m \times n \) matrix such that \( A^T A \) is invertible. Show that the columns of \( A \) are linearly independent. (Note: We can’t assume that \( A \) is invertible; it may not even be square.)