Worksheet 3 Solutions, Math 1B
Integration by Partial Fractions, Other Substitutions

Monday, January 30, 2012

1. Use the Weierstrass substitution to find the indefinite integral of \( \sec(x) \). Use trigonometric identities to show that this expression is equivalent to the one derived in class.

Solution Sketch
The result after the substitution and integration by parts is

\[
\ln \left| \frac{1 + \tan \left( \frac{x}{2} \right)}{1 - \tan \left( \frac{x}{2} \right)} \right|.
\]

Multiplying by \( 1 + \tan \left( \frac{x}{2} \right) \) in both the numerator and denominator of the trig expression and splitting gives

\[
\frac{1 + \tan \left( \frac{x}{2} \right)}{1 - \tan \left( \frac{x}{2} \right)} = \frac{2 \tan \left( \frac{x}{2} \right)}{1 - \tan^2 \left( \frac{x}{2} \right)} + \frac{1 + \tan^2 \left( \frac{x}{2} \right)}{1 - \tan^2 \left( \frac{x}{2} \right)},
\]

and rewriting the second expression in terms if sines and cosines gives

\[
\frac{\tan \left( \frac{x}{2} \right)}{1 - \tan^2 \left( \frac{x}{2} \right)} = \frac{1}{\cos^2 \left( \frac{x}{2} \right) - \sin^2 \left( \frac{x}{2} \right)}.
\]

But then double angle formulas for cosine and tangent give

\[
\frac{2 \tan \left( \frac{x}{2} \right)}{1 - \tan^2 \left( \frac{x}{2} \right)} + \frac{1}{\cos^2 \left( \frac{x}{2} \right) - \sin^2 \left( \frac{x}{2} \right)} = \tan(x) + \sec(x),
\]

and this is the form derived in class.

2. Evaluate the following integrals:

(a) \( \int \frac{x^3}{x^3 + 1} \, dx \)

Solution Idea
Rewrite the integral as

\[
\int \frac{x^3}{x^3 + 1} \, dx = \int 1 - \frac{1}{x^3 + 1} \, dx = x - \int \frac{1}{(x + 1)(x^2 - x + 1)} \, dx,
\]

and integrate by parts as usual.

(b) \( \int \frac{x^3 + 4}{x^2 + 4} \, dx \)

Solution Idea
Rewrite the integral as

\[
\int \frac{x^3 + 4}{x^2 + 4} \, dx = \int x - \frac{4x - 4}{x^2 + 4} \, dx = \int x \, dx - 2 \int \frac{2x}{x^2 + 4} \, dx + \int \frac{1}{1 + \left( \frac{x}{2} \right)^2} \, dx,
\]

and each of these is solvable using previous techniques.
(c) \[ \int \frac{1}{x \sqrt{x + 1}} \, dx \]

**Solution Sketch**

This integral only makes sense for \( x > -1 \), so we assume that this is the case. First use a substitution such that \( x + 1 = u^2 \), namely, \( u = \sqrt{x + 1} \), with \( dx = 2udu \) and \( x = u^2 - 1 \). Then

\[
\int \frac{1}{x \sqrt{x + 1}} \, dx = \int \frac{2u}{(u^2 - 1)u} \, du = 2 \int \frac{u}{(u + 1)(u - 1)},
\]

and this we can solve as usual with integration by parts.

(d) \[ \int \frac{x^3}{\sqrt{x^2 + 1}} \, dx \]

**Solution Sketch**

First use a substitution such that \( x^2 + 1 = v^3 \), namely, \( v = \sqrt{x^2 + 1} \). In this case, \( 2x \, dx = 3v^2 \, dv \), and in particular, we can write

\[
x^3 \, dx = \frac{x^2(2x \, dx)}{2} = \frac{(x^2 + 1) - 1)(2x \, dx)}{2} = \frac{(v^3 - 1)(3v^2 \, dv)}{2},
\]

and the integral becomes

\[
\frac{3}{2} \int v(v^3 - 1) \, dv.
\]

3. The functions \( y = e^{x^2} \) and \( y = x^2 e^{x^2} \) don’t have elementary antiderivatives, but \( y = (2x^2 + 1)e^{x^2} \) does. Evaluate

\[ \int (2x^2 + 1)e^{x^2} \, dx \]

**Solution**

We write the integral as

\[ \int e^{x^2} \, dx + \int 2x^2 e^{x^2}, \]

and use integration by parts on the first integral, with \( u = e^{x^2} \) and \( dv = dx \). This gives

\[ \int e^{x^2} \, dx = xe^{x^2} - \int 2x^2 e^{x^2}, \]

and as a consequence, we see that the overall integral has value \( xe^{x^2} \).

4. Factor \( x^4 + 1 \) as a difference of squares by first adding and subtracting the same quantity. Use this factorization to evaluate

\[ \int \frac{1}{x^4 + 1} \, dx \]

**Solution Idea**

The suggested method is a variant on completing the square, where instead of adding and subtracting a constant value, we add and subtract an intermediate value:

\[
x^4 + 1 = x^4 + 2x^2 + 1 - 2x^2 = (x^2 + 1)^2 - (\sqrt{2}x)^2 = (x^2 + \sqrt{2}x + 1)(x^2 - \sqrt{2}x + 1).
\]

The quadratic formula reveals that both of these polynomials are irreducible, and so we can continue by using partial fractions with this factorization.