1. Evaluate the integral
\[ \int \frac{x^3}{\sqrt{4 + x^2}} \, dx \]
using trigonometric substitution. Then evaluate it using integration by parts.

Solution Idea
Trigonometric substitution calls for a tangent-type substitution. Integration by parts makes use of the
initial factorization \( u = x^2 \) and \( dv = x/\sqrt{4 + x^2} \, dx \).

2. Evaluate the following integrals:
(a) \[ \int_0^1 x \sqrt{x^2 + 4} \, dx \]

Solution
We use a tangent-type substitution, \( x = 2 \tan(\theta) \) (where \(-\pi/2 < \theta < \pi/2\)), which gives us
\( dx = 2 \sec^2(\theta) \, d\theta \), and
\[
\int_0^1 x \sqrt{x^2 + 4} \, dx = \int_{x=0}^{x=1} \left(2 \tan(\theta)\right) \sqrt{(2 \tan(\theta))^2 + 4} \left(2 \sec^2(\theta)\right) \, d\theta
\]
\[ = 8 \int_{\theta=0}^{\theta=1} \tan(\theta) \sqrt{\tan^2(\theta) + 1} \sec^2(\theta) \, d\theta \]
\[ = 8 \int_{\theta=0}^{\theta=1} \tan(\theta) \sec^2(\theta) \sec^2(\theta) \, d\theta = 8 \int_{\theta=0}^{\theta=1} \tan(\theta) \sec^3(\theta) \, d\theta \]

From this we can use a simple substitution \( u = \sec(\theta) \) to find a value of
\[ \left(\frac{8 \sec^3(\theta)}{3}\right) \bigg|_{x=0}^{x=1}. \]

Using a right triangle to represent the substitution \( x = 2 \tan(\theta) \), we find that \( \sec(\theta) = \sqrt{x^2 + 4}/2 \),
and so the final result is
\[ \left(\left(x^2 + 4\right)^{3/2}/3\right) \bigg|_{x=0}^{x=1} = (\sqrt{125} - 8)/3. \]

Alternatively, a simpler solution uses substitution with \( u = x^2 + 4 \).
(b) \( \int \sqrt{5 + 4x - x^2} \, dx \)

**Solution Idea**
Completing the square yields
\[
\int \sqrt{5 + 4x - x^2} \, dx = \int \sqrt{9 - (x - 2)^2} \, dx,
\]
which calls for a sine-type substitution after first substituting \( u = x - 2 \).

(c) \( \int_{0}^{a} x^2 \sqrt{a^2 - x^2} \, dx \)

**Solution Sketch**
A first quick substitution of \( u = ax \) gives
\[
\int_{0}^{a} x^2 \sqrt{a^2 - x^2} \, dx = a^4 \int_{0}^{1} u^2 \sqrt{1 - u^2} \, du,
\]
and from here a sine-type substitution gives us an integral of the form
\[
\int \sin^2(\theta) \cos(\theta) \, d\theta.
\]
Several applications of the half-angle formulas allow a solution.

(d) \( \int \frac{x^2}{9 - 25x^2} \, dx \)

**Solution Idea**
A substitution of \( x = (3/5) \sin(\theta) \) makes this tractable.

(e) \( \int \frac{1 - \tan^2 x}{\sec^2 x} \, dx \)

**Solution Idea**
Converting into sines and cosines and simplifying shows that this integral is actually equal to
\[
\int \cos(2x) \, dx.
\]

(f) \( \int_{0}^{\pi/2} \frac{\cos t}{\sqrt{1 + \sin^2 t}} \, dt \)

**Solution Idea**
Make an initial substitution of \( u = \sin(t) \) to turn this into a usual trig substitution integral, which is amenable to a tangent-type substitution.
3. A torus is generated by rotating the circle \( x^2 + (y - R)^2 = r^2 \) about the \( x \)-axis. Find the volume enclosed by the torus.

**Solution Idea**

The cross-section of the circle at a given value of \( x \in [-r, r] \) goes from \( R - \sqrt{r^2 - x^2} \) to \( R + \sqrt{r^2 - x^2} \), and so the cross-sectional area of the solid of revolution is given by

\[
\pi (R + \sqrt{r^2 - x^2})^2 - \pi (R - \sqrt{r^2 - x^2})^2.
\]

Then the overall volume is an integral over all the values of \( x \) of the cross-sectional area, which is

\[
V = \int_{-r}^{r} \pi (R + \sqrt{r^2 - x^2})^2 - \pi (R - \sqrt{r^2 - x^2})^2 = 4\pi R \int_{-r}^{r} \sqrt{r^2 - x^2}.
\]

This can be solved using a simple sine-style trig substitution, but alternatively we can see that the integral is equal to the area above the \( x \)-axis of the circle of radius \( r \) centered at the origin, which is \((\pi r^2)/2\). This gives us a final volume of \( 2\pi R r^2 \).