Evaluate the following integrals (5 pts each):

1. $\int x^2 \cos(x) \, dx$

   **Solution**

   We apply integration by parts twice. First we use $u = x^2$ and $dv = \cos(x) \, dx$, which gives us $du = 2x \, dx$ and $v = \sin(x)$, and gives us an equivalent expression of

   \[ x^2 \sin(x) - \int 2x \sin(x) \, dx. \]

   For the second time, we use $u = x$ and $dv = -\sin(x) \, dx$, which gives us $du = dx$ and $v = \cos(x)$. The resulting expression is

   \[ x^2 \sin(x) + 2x \cos(x) - \int \cos(x) \, dx = (x^2 - 2) \sin(x) + 2x \cos(x) + C. \]

2. $\int \tan^3(x) \sec(x) \, dx$

   **Solution**

   We first apply the Pythagorean identity $1 + \tan^2(x) = \sec^2(x)$ to get

   \[ \int (\sec^2(x) - 1) \tan(x) \sec(x) \, dx. \]

   From here we can simply substitute using $u = \sec(x)$, to get

   \[ \int (u^2 - 1) \, du = u^3/3 - u + C = \sec^3(x)/3 - \sec(x) + C. \]
3. \( \int_0^a \sqrt{a^2 - x^2} \, dx \)

**Solution**

We begin by making a sine-type substitution \( x = a \sin(\theta) \), where \( \theta \) is chosen between \(-\pi/2\) and \(\pi/2\). Under this substitution, the integral becomes

\[
\int_0^{\pi/2} \sqrt{a^2 - a^2 \sin(\theta) a \cos(\theta)} \, d\theta,
\]

and using the Pythagorean identity \( 1 - \sin^2(\theta) = \cos^2(\theta) \), we see that this is just

\[
a^2 \int_0^{\pi/2} \cos^2(\theta) \, d\theta.
\]

To solve this, we need to make use of the half-angle formula

\[
\cos^2 x = \frac{1 + \cos 2x}{2}
\]

to rewrite this integral as

\[
a^2 / 2 \int_0^{\pi/2} (1 + \cos(2\theta)) \, d\theta = a^2 / 2(\theta + \sin(2\theta)/2) \bigg|_0^{\pi/2} = a^2 / 2(\pi/2 + 0 + 0 + 0) = \pi a^2 / 4.
\]

If the bounds of integration were not translated with the substitution \( x = a \sin(\theta) \), the reverse substitution would also require the use of the double-angle formula \( \sin(2\theta) = 2 \sin(\theta) \cos(\theta) \) along with the usual triangle construction for combining trig and inverse trig functions.

Notice that this solution makes sense, as the integral is over the upper-right quadrant of a circle of radius \( a \) centered at the origin, and so its value should be equal to a quarter of the circle’s area.