1. Evaluate the following integral:
\[
\int \frac{2x^2 + 1}{x(x - 1)^2} \, dx
\]

Solution

We use partial fractions to simplify the integral. The partial fractions expansion is of the form
\[
\frac{2x^2 + 1}{x(x - 1)^2} = \frac{A}{x} + \frac{B}{x - 1} + \frac{C}{(x - 1)^2}
\]

Equating coefficients of respective powers of \(x\) and solving the resulting system of equations gives values \(A = 1\), \(B = 1\), and \(C = 3\), so
\[
\int \frac{2x^2 + 1}{x(x - 1)^2} \, dx = \int \frac{1}{x} + \frac{1}{x - 1} + \frac{3}{(x - 1)^2} \, dx
\]
\[
= \int \frac{1}{x} \, dx + \int \frac{1}{x - 1} \, dx + \int \frac{3}{(x - 1)^2} \, dx
\]
\[
= \ln|x| + \ln|x - 1| - \frac{3}{x - 1} + C
\]

2. Write out, but do NOT simplify, the midpoint approximation \(M_4\) of
\[
\int_0^2 x \sin(x) + \cos(x) \, dx.
\]

Calculate an error bound for this approximation using the Midpoint Rule error bound equation
\[
|E_M| \leq K(b - a)^3/24n^2.
\]
Solution

If we write \( f(x) = x \sin(x) + \cos(x) \), then the midpoint approximation \( M_4 \) with \( n = 4 \) is given by

\[
\frac{2 - 0}{4} \left( f(1/4) + f(3/4) + f(5/4) + f(7/4) \right).
\]

To calculate the error bound, we need to find an upper bound on the second derivative of \( f \). We calculate

\[
f'(x) = x \cos(x) + \sin(x) - \sin(x) = x \cos(x),
\]

and

\[
f''(x) = \cos(x) - x \sin(x),
\]

and we estimate

\[
|f''(x)| = |\cos(x) - x \sin(x)| \leq |\cos x| + |-x \sin(x)|
\]

\[
= |\cos x| + |x| |\sin x| \leq 1 + 2 \cdot 1 = 3.
\]

Thus we can use \( K = 3 \) as a bound on the second derivative, and this gives us an error estimate of

\[
|E_M| \leq 3(2 - 0)^3/(24 \cdot 4^2) = 1/16.
\]

3. Use the comparison theorem to determine whether the following improper integral is convergent or divergent:

\[
\int_1^\infty \frac{e^{-x}}{\cos(x)/2 + 1} \, dx
\]

Solution

We use the comparison theorem to show that the integral is convergent. Because \( \cos(x) \) only varies between -1 and 1, we see that \( \cos(x)/2 + 1 \) only varies between 1/2 and 3/2, and in particular is at least 1/2 for every \( x \). Thus

\[
0 \leq \frac{e^{-x}}{\cos(x)/2 + 1} \leq 2 e^{-x}.
\]

We can calculate

\[
\int_1^\infty e^{-x} = \lim_{s \to \infty} \int_1^s e^{-x} = \lim_{s \to \infty} (-e^{-s} + e^{-1}) = 1/e,
\]

which is convergent. Thus by the comparison theorem, we see that the given improper integral is also convergent.