MONOPOLE DYNAMICS AND HYPERKÄHLER GEOMETRY

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1. Physical background

First, let’s recall how gauge theories work. The simplest of these is electromagnetism, which has gauge group $U(1)$. Mathematically this means that we are studying a principal $U(1)$-bundle with connection $A$ over spacetime (in general, a Riemannian manifold; here, let’s stick to $\mathbb{R}^4$). Since $\mathbb{R}^4$ is simply-connected and $u(1) \cong \mathbb{R}$ is an abelian Lie algebra, the theory is very simple: the bundle must be trivial, and a connection is just a 1-form with values in $u(1) \cong \mathbb{R}$, or in other words an ordinary one-form: $A \in \Omega^1(M)$. The curvature of $A$, in general, is $F_A = dA + A \wedge A$. The Maxwell action is given by $\int_M F_A \wedge * F_A$, so the equations of motion are:

$$dF_A = 0 = d * F_A,$$

where $*$ is the Hodge star coming from the metric on $M$.

But this isn’t very exciting; we want to consider non-abelian gauge theories. So now take exactly the above situation, but with $U(1)$ replaced by $G = SU(2)$. (We could take $G = SU(N)$, but we’ll only really care about $SU(2)$ here. The Lie algebra $su(2)$ is non-abelian, but otherwise the situation is pretty similar; the only real difference is now our equations of motion use the covariant derivative

$$d_A F_A = 0 = d_A * F_A.$$

So far this theory gives us massless particles: a massive particle requires a term like $m^2 Q(A)$ in the action. To get this, we need another field $\phi$ in our theory, which mathematically is a section of the adjoint bundle $ad(P) = P \times_G g$. Then we can take our action to be $a = (F, F) + (d_A \phi, d_A \phi) + V(\phi)$ for $V$ a gauge-invariant potential function. In practice we will take $V \equiv 0$. Then our equations of motion become

$$d_A F = 0, \quad d_A * F = -[\phi, d_A \phi].$$

Why are these monopoles? Remember that we are looking for finite action solutions of these equations which are localized in space (i.e., which look like particles). The finite action conditions requires that in the "$S^2$ at infinity" of space, our field is "pure gauge," i.e., zero curvature: the connection at any point is $g^{-1} dg$. So each solution gives us a map $S^2 \to SU(2)/U(1)$. Since this group has $\pi_2 = \mathbb{Z}$, the space of solutions has $\mathbb{Z}$ many distinct path-components, corresponding to the magnetic charge.

Intuition for why there should be magnetic charge: For a finite energy solution, the term $d_A \phi$ must decay in $1/r^2$. If $\phi$ varies around $S^2_\infty$, then $d\phi \sim 1/r$, so to cancel this term we need to turn on a gauge potential $A_\theta \sim 1/r$, giving a magnetic field $B \sim 1/r^2$.

Okay, that’s most of the physics out of the way. The key insight of Manton is that we can pretend this is a mechanical problem: Let $\mathcal{A}$ be the space of pairs $(A, \phi)$ on $\mathbb{R}^3$ with appropriate decay at $\infty$, $\mathcal{G}$ the group of gauge transformations, and $\mathcal{C} := \mathcal{A}/\mathcal{G}$. Then

$$T_{(A, \phi)} \mathcal{C} = \{(\dot{A}, \dot{\phi}) \in \mathcal{A} | d^*_A \dot{A} + [\phi, \dot{\phi}] = 0\}$$

and we can write down a metric

$$h(\dot{c}, \dot{c}) = \int_{\mathbb{R}^3} (\dot{A}, \dot{A}) + (\dot{\phi}, \dot{\phi}),$$

and a potential function

$$U = \frac{1}{2} \int_{\mathbb{R}^3} (F_A, F_A) + (d_A \phi, d_A \phi).$$
Now we pretend our monopole is a particle moving on the infinite-dimensional manifold $C$, evolving in time according to the potential $U$. Let’s calculate this integral over a ball of radius $R$. Using Bogomolny’s trick of completing the square, we find
\[
\int_{B_R} (F,F) + (d_A \phi, d_A \phi) = \int_{B_R} (F - *d_A \phi, F - *d_A \phi) + 2(*d_A \phi, F).
\]
The second term becomes $\int_{S_R} (\phi, F)$, and at $R \to \infty$ this becomes $\pm 4\pi k$, where $k$ is the winding number discussed earlier, i.e., the Chern class of the eigenspace associated to $\phi$, which is a line bundle over $S_R$. (This is due to the asymptotic conditions $|d_A \phi| = O(r^{-2}), |\phi| \to 1$; therefore we find that
\[
\int_{R^3} (F,F) + (d_A \phi, d_A \phi) = \int_{R^3} |F - *d_A \phi|^2 \pm 8\pi k.
\]
Hence the absolute minimum of $U$ is $4\pi k$, occurring when $(A, \phi)$ satisfy the Bogomolny equations
\[
F = *d_A \phi.
\]

2. Rational maps

Let’s remind ourselves of what we’ve got. We are looking for connections and Higgs fields on $\mathbb{R}^3$ that satisfy the Bogomolny equations, with asymptotic assumptions
\[
|\phi| = 1 - \frac{k}{2r} + O(r^{-2})
\]
\[
\frac{\partial |\phi|}{\partial \Omega} = O(r^{-2})
\]
\[
|D\phi| = O(r^{-2}).
\]
Let $N_k$ be the moduli space of such solutions. We will actually consider an extension of this space by $S^1$ called the moduli space of framed monopoles, defined as follows: pick a direction (we’ll choose $x_1$), use the $A_1 = 0$ gauge, and only allow gauge transformations which tend to the identity as $x_1 \to \infty$. We’ll call this space $M_k$ (or $M_k(x_1)$ if we want to remember the dependence on our choice). The first major theorem we’ll highlight is due to Donaldson:

**Theorem 2.1** (Donaldson). $M_k$ is diffeomorphic to the space $R_k$ of all degree $k$ rational functions which are $0$ at $\infty$.

It was later realized (by Hurtubisse) that the study of scattering and the machinery of the spectral curve give a neat description of this diffeomorphism.

We will get information about our moduli space by studying the scattering associated to the operator
\[
D_u := \nabla_u - i\phi,
\]
where $\nabla$ is the covariant derivative on our $SU(2)$ bundle $E$, and $u$ is a geodesic in $\mathbb{R}^3$ (i.e., an oriented straight line). Hitchin proved that the differential equation $D_u s = 0$ has two linearly independent solutions, which we can choose so that as $t \to \infty$ we have
\[
s_0(t)t^{-k/2}e^t \to e_0
\]
\[
s_1(t)t^{k/2}e^{-t} \to e_1,
\]
where $e_0, e_1$ form a basis for $E$ at $\infty$ (i.e., they are constant in the asymptotic gauge). Note that $s_0(t)$ has exponential decay at infinity. Similarly, we can find a solution $s'_0$ which has exponential decay at $-\infty$. Let us write this in terms of our basis as
\[
s'_0 = as_0 + bs_1.
\]
Then define
\[
S(u) := \frac{a}{b}.
\]

Now, in order to mimic the way we defined $M_k(x_1)$, let’s fix an isomorphism $\mathbb{R}^3 \cong \mathbb{R} \times \mathbb{C}$, let $u(z)$ be the line parallel to the $x_1$-axis through $(0, z)$. Since elements $m \in M_k(x_1)$ are trivialized at $x_1 = \infty$, we get a well-defined map $z \mapsto S_m(z) := (u(z))$. This map is precisely the diffeomorphism $M_k(x_1) \to R_k$ which we were looking for.
There are a couple of remarks which could be made here. First, note that the poles of \( S \) are precisely those lines for which \( b = 0 \), i.e., \( s_b' = as_0 \), i.e., \( s_0 \) has exponential decay at both \( \infty \) and \(-\infty \) along the line \( u \). Let us denote the set of all such lines by \( \Gamma \). \( \Gamma \) is an algebraic curve in the space of all straight lines in \( \mathbb{R}^3 \), which is isomorphic to \( TP^1 \) (we can parametrize it by directions (points of \( S^2 \)) and translations orthogonal to the \( S^2 \) radial vector), and we call \( \Gamma \) the spectral curve associated to our monopole.

Second, it turns out that this theorem has a generalization to any compact semisimple group, proved in 1996 by Donaldson’s student Stuart Jarvis.

**Theorem 2.2** (Jarvis). *Choosing a framing of the moduli space of \( G \)-monopoles of charge \( \alpha \in H_2(G, \mathbb{Z}) = \Lambda \) gives an isomorphism between this space and the space of degree \( \alpha \) based rational maps \( \mathbb{P}^1 \to G/B \).*

This latter space is of great interest in many areas of mathematics I will return to this if I have time, but I will point out one fun fact now in the \( SU(2) \) case: Donaldson’s theorem tells us that our monopole space is isomorphic to the space of all rational functions of the form

\[
S(z) = \frac{\sum_{i=1}^{k-1} a_i z^i}{z^k + \sum_{i=1}^{k} b_i z^i},
\]
satisfying the condition that the numerator and denominator are coprime, which is witnessed by the nonvanishing of the resultant \( \Delta(a, b) \), so we have \( M_k \cong \mathbb{C}^{2k} \setminus \Delta(a, b) \). One partial compactification we can perform is replacing \( \Delta(a, b) \) back into the space, to get \( M_k \cong \mathbb{C}^{2k} \). This space \( M_k \) is Drinfeld’s compactification of Zastava, i.e., the space of quasimaps to \( G/B = \mathbb{P}^1 \). This space plays a key rôle in the study of quantum K-theory of flag varieties.

### 3. Hyperkähler metrics

Now let’s investigate a few different ways of seeing the hyperkähler structure on this moduli space. We’ll want to look at the tangent space, so let’s pick a charge \( k \) monopole \( \mp(A, \phi) \), and let \( T_c \) the set of all \( (a, \psi) \) which are \( L^2 \) and satisfy:

\[
* d_A a - d_A \psi + [\phi, a] = 0
\]

\[
* d_A a + [\phi, \psi] = 0.
\]

These are, respectively, the linearization of the Bogomolny equation and a stipulation that \( (a, \psi) \) is orthogonal to directions of compactly supported infinitesimal gauge transformations. Some analytical results of Taubes prove that this is in fact the tangent space of \( M_k \) at \( c \). [N.B.: This doesn’t always happen. In some situations, not all directions can be represented by \( L^2 \) variations.]

One easy way of seeing a hyperkähler structure on \( M_k \) is by noting that the space of all \( (a, \psi) \) can be identified with the space of functions to \( \text{su}(2) \otimes \mathbb{H} \). Explicitly, if \( \alpha = \alpha dx + \beta dy + \gamma dz \), then \( (a, \psi) \) corresponds to \( \psi + \alpha I + \beta J + \gamma K \). But now, note that the equations above are \( \mathbb{H} \)-invariant, which means that \( T_c M_k \) is an \( \mathbb{H} \)-vector space, and so \( M_k \) possesses an almost quaternionic structure. But Donaldson’s theorem identifies \( M_k \) with the complex manifold \( R_k \), and the identification (and hence the complex structure) depends on a direction, i.e., a point in \( S^2 \), i.e., a unit quaternion!

Here’s a good way of looking at it: choosing an axis \( x_1 \) in \( \mathbb{R}^3 \) chooses an isomorphism \( \mathbb{R}^3 \times S^1 \cong \mathbb{C} \times \mathbb{C}^* \) (by \( (\vec{x}, \theta) \mapsto (x_2 + i x_3, \exp(x_1 + 2\pi i \theta)) \)). A monopole gives an \( S^1 \)-invariant solution of \( F = -* F \) on \( \mathbb{R}^3 \times S^1 \) and thus corresponds to a holomorphic bundle on \( \mathbb{C} \times \mathbb{C}^* \) with \( \mathbb{C}^* \) action. The asymptotic conditions are that the bundle is trivialized at \( 0 \) and \( \infty \) in \( \mathbb{C}^* \), and the associated scattering is precisely the transition function at \( 1 \in \mathbb{C}^* \). Basically, Donaldson’s theorem relates the description of a holomorphic line bundle through its \( \bar{\partial} \) operator to the description through transition functions.

Anyway, using \( I, J, K \), we can define 2-forms \( \omega_1, \omega_2, \omega_3 \), for instance:

\[
\omega_1((a_1, \psi_1), (a_2, \psi_2)) = \int_{\mathbb{R}^3} (I(a_1, \psi_1), (a_2, \psi_2))
\]

\[
= \int_{\mathbb{R}^3} (\psi_1, a_2) - (a_1, \psi_2) + (\beta_1, \gamma_2) - (\gamma_1, \beta_2).
\]

These are constant, hence closed, and we can check that these actually descend to the tangent space of \( M_k \).

We might also notice that our moduli space is a hyperkähler quotient: Let \( V \) be the space of all \( (a, \psi) \) asymptotically of order \( r^{-2} \), and let \( G \) be the group of all gauge transformations of order \( r^{-1} \). Then the
Bogomolny equations are precisely the triple \((\mu_1, \mu_2, \mu_3)\) of hyperkähler moment maps we get from the three symplectic structures, and \(\bigcap \mu_i^{-1}(0)/G\) is the moduli space.

The hyperkählerity of the monopole space allows us to describe its metric using twistor constructions:

**Definition 3.1.** Let \(M\) be a Kähler 4-manifold. The **twistor space** of \(M\) is the manifold \(Z\) equipped with the integrable almost complex structure \(I = (aI + bJ + cK, I_0)\) at the point \((m, a, b, c)\).

Choose a complex coordinate \(\zeta\) on \(S^2 \cong \mathbb{P}^1\). Then we have a complex 2-form \(\omega = (\omega_2 + i\omega_3) - 2\omega_1\zeta - (\omega_2 - i\omega_3)\zeta^2\) is a holomorphic section of \(\Lambda^2 T^*_F \otimes O(2)\). \(Z\) also possesses an antiholomorphic involution \(\tau: Z \to Z\) given by the antipodal map on \(S^2\).

**Theorem 3.2.** Let \(Z^{2n+1}\) be a complex manifold. Then it is a twistor space for a hyperkähler manifold \(M^{4n} := \Gamma(\mathbb{P}^1, Z)\) if and only if:

(i) \(Z \to \mathbb{P}^1\) is a holomorphic fibre bundle,
(ii) admitting a family of holomorphic sections with normal bundle \(\mathbb{C}^{2n} \otimes O(1)\);
(iii) with a symplectic form on each fibre, given by a holomorphic section \(\omega\) of \(\Lambda^2 T^*_F \otimes O(2)\),
(iv) and a real structure \(\tau\) which is antipodal map on \(\mathbb{P}^1\).

The twistor space can be used to derive the Kähler metric on \(M\). This procedure works especially well when \(M\) admits an \(SO(3)\) action which moves around its complex structures. In this case, take the \(SO(2) \subset SO(3)\) which fixes \(\omega_2\), which is generated by a vector field \(X\). We can get \(X\) by looking at the copy of \(M\) sitting above \(\zeta = 0\). This fibre has complex structure \(\omega_2 + i\omega_3\), and \(\mathcal{L}_X(\omega_2 + i\omega_3) = i\omega_1\), so we know the Kähler form and hence the metric. The monopole spaces \(M_k\) admit nice twistor spaces which are very useful.

4. **The simplest examples:** \(k = 1, 2\)

I didn’t want to conclude without writing down an example of a monopole. So here is one (in fact, here is all charge 1 monopoles): \(S(z) = \frac{a}{z^3}\). The standard monopole, centered at the origin, is \(S(z) = \frac{1}{z}\). The general monopole is located at \((-\frac{1}{2}\log|a|, b)\) with phase \(\arg(a)\). Thus \(M_1 \cong \mathbb{C} \times \mathbb{C}^*\). More generally, pick \(S \in R_k\) such that \(S\) has simple poles:

\[ S(z) = \sum_{i=1}^{k} \frac{a_i}{z - b_i}. \]

It turns out that as the \(b_i\) move away from each other, this monopole approaches a combination of \(k\) simple monopoles with very little interaction.

One last example: for \(k = 2\) consider the manifold \(M_2^0\) which we have quotiented by the action of translations. Our rational functions are of the form \(S(z) = \frac{z^3 + a_1 z}{z^2 + b_0}\), and the resultant is \(\Delta = z_0^2 + b_0a_1^2\). So setting \(\Delta = 1\) gives a connected double cover of \(M_2^0\): this is the surface in \(\mathbb{C}^3\) cut out by \(x^2 - y^2z^2 = 1\). Finally, \(M_2^0\) is the quotient of this space by the involution \((x, y, z) \mapsto (-x, -y, z)\).

**References**