Math 274: Tropical Geometry

UC Berkeley, Spring 2009
Homework # 5, due Thursday, March 12

1. Show that every balanced one-dimensional fan in $\mathbb{R}^3$ is the tropicalization of a space curve in $\mathbb{C}^3$. More precisely, let $v_1, v_2, \ldots, v_n$ be integer vectors in $\mathbb{Z}^3$ such that $v_1 + v_2 + \cdots + v_n = 0$, and let Σ be the union of the n rays $\mathbb{R}_{\geq 0} v_i$. Construct a prime ideal $I \subset \mathbb{C}[x, y, z]$ with $\mathcal{T}(I) = \Sigma$.

2. Determine the maximal Barvinok rank of any 5×5-matrix whose entries are 0 or 1. Do you have a conjecture for $n \times n$-matrices over $\{0, 1\}$?

3. A *polytrope* is a subset P of tropical projective space $\mathbb{TP}^d$ such that P is both tropically convex and classically convex. What is the maximal number of classical vertices of 4-dimensional polytrope ($d = 4$)?

4. Consider a general arrangement of n tropical hyperplanes in $\mathbb{TP}^d$. How many components are there in the complement of this arrangement ?

5. Consider the family of plane cubic curves of the special form

$$c_1 x^2 y + c_2 x y^2 + c_3 x^2 + c_4 x y + c_5 y^2 + c_6 x + c_7 y \quad = \quad 0.$$

   (a) Compute the Newton polytope of the discriminant of this family.
   (b) How many rational curves of the above special form pass through five general points in the complex projective plane?
   (c) Draw a picture that demonstrates the tropical solution to this curve counting problem, i.e., pick five general points in the plane $\mathbb{TP}^2$ and determine all tropical curves of genus zero passing through your points.

6. Write a short essay ($\leq$ 2 pages) on one of the following three questions:

   - *What is a "tropical vertex" ?* (Mark Gross' MSRI talk on Feb 23).
   - *How are cluster algebras related to tropical geometry ?*
   - *Is every balanced graph in* 3*-space a tropicalized space curve ?*