Math 113, **Final Exam**
SOLUTIONS

(1) The characteristic of the field $K = \mathrm{GF}(25)$ is 5. Thus, every non-zero element of K has order 5 in the additive group $(K, +)$, and hence this group is isomorphic to $\mathbf{Z}_5 \times \mathbf{Z}_5$. The multiplicative group $(K^*, \cdot)$ consists of all non-zero elements in K, so it has order 24. By Theorem 33.5, this group is cyclic, and hence it is isomorphic to $\mathbf{Z}_{24}$.

(2) This problem is similar to # 14 on page 197. The ring $Q(R, T)$ is constructed by starting with the set of pairs $R \times T = \{0, 1, 2, 3, 4, 5\} \times \{1, 5\}$, and then forming the classes of the equivalence relation $(r, t) \sim (r', t')$ defined by $rt' = r't$. There are **six** classes

$$\frac{0}{1} = \{(0,1),(0,5)\}, \quad \frac{1}{1} = \{(1,1),(5,5)\}, \quad \frac{2}{1} = \{(2,1),(4,5)\},$$
$$\frac{3}{1} = \{(3,1),(3,5)\}, \quad \frac{4}{1} = \{(4,1),(2,5)\}, \quad \frac{5}{1} = \{(1,5),(5,1)\}.$$

From this we see that $Q(R, T)$ is isomorphic to $R = \mathbf{Z}_6$.

(3) The symmetry group of the square (with vertices $1, 2, 3, 4$) is the dihedral group D_4 which has order 8. For each $g \in D_4$ we list number of colorings that are fixed under g:

g	id	(1234)	(1432)	(13)(24)	(13)	(24)	(12)(34)	(14)(23)
$\lvert X_g \rvert$	n^4	n	n	n^2	n^3	n^3	n^2	n^2

Burnside's Formula tells us that the number of colorings is

$$\frac{1}{8}\sum_{g \in D_4} X_g \;=\; \frac{1}{8}\big(n^4 + 2n^3 + 3n^2 + 2n\big) \;=\; \frac{1}{8} n(n+1)(n^2+n+2).$$

For $n = 6$, this number equals 231, as seen in Exercise # 7 (b) on page 231.

(4) Each additive group homomorphism $\phi : \mathbf{Z} \times \mathbf{Z} \times \mathbf{Z} \to \mathbf{Z}$ is uniquely determined by its value on the three generators $e_1 = (1,0,0)$, $e_2 = (0,1,0)$ and $e_3 = (0,0,1)$. Moreover, $a_i = \phi(e_i)$ has to be equal to 0 or 1 in order for ϕ to be a ring homomorphism since $a_i = \phi(e_i \cdot e_i) = \phi(e_i) \cdot \phi(e_i) = a_i^2$. However, we must have $a_i a_j = 0$ for $1 \le i < j \le 3$ since

$$a_i + a_j = \phi\big(e_i + e_j\big) = \phi\big((e_i+e_j)(e_i+e_j)\big) = \phi(e_i+e_j)\phi(e_i+e_j) = (a_i+a_j)^2 = a_i + a_j + 2a_i a_j.$$

We conclude that there are precisely **four** ring homomorphisms ϕ. They are given by

$$(a_1, a_2, a_3) \in \big\{\, (0,0,0)\,,\ (1,0,0)\,,\ (0,1,0)\,,\ (0,0,1) \,\big\}.$$

(5) The sequence $\{\mathrm{id}\} < A_3 < S_3$ is a composition series because the two factor groups are cyclic of prime order, so they are simple. The following is a composition series for $S_3 \times S_3$:

$$\{\mathrm{id}\} \times \{\mathrm{id}\} \;<\; \{\mathrm{id}\} \times A_3 \;<\; \{\mathrm{id}\} \times S_3 \;<\; A_3 \times S_3 \;<\; S_3 \times S_3.$$

The consecutive factor groups are cyclic of order 2 or 3, so they are simple and abelian. By Definition 35.18, this means that the group $S_3 \times S_3$ is solvable.

(6) This is the special case $n = 4$ of Exercise # 39 on page 96. We consider the subgroup of S_4 generated by the transposition (12) snd the 4-cycle (1234). That subgroup contains

$$(23) = (1234)(12)(1234)^3, \; (34) = (1234)^2(12)(1234)^2, \; (14) = (1234)^3(12)(1234)$$

and hence also

$$(13) \; = \; (23)(12)(23) \quad \text{and} \quad (24) \; = \; (23)(34)(23).$$

So, we see that this subgroup contains all six transpositions. But S_4 is generated but its transpositions, by Corollary 9.12. Therefore the group S_4 is generated by (12) and (1234).

(7) We apply the Sylow Theorems to show that every group G of order $96 = 2^5 3$ has a proper normal subgroup. The argument is analogous to that in Example 37.13 on page 331. The number of Sylow 2-subgroups is odd and divides 96, so it is either 1 or 3. If it equals 1 then the unique Sylow 2-subgroup is a normal of order 32 in G, and we are done. So, we assume that there are 3 subgroups of order 32. Let H and K be two of them. Then the order of $H \cap K$ must equal 16; for, otherwise if $|H \cap K| \leq 8$ then HK has order at least $\frac{32 \cdot 32}{8} = 128$ by Lemma 37.8, and this would exceed $|G| = 98$. Now, since $H \cap K$ has index 2 in H, it is normal in H, and, similarly, it is normal in K. The normalizer is a subgroup of G that properly contains both H and K, so its order is a proper multiple of 32 and it divides 96. This implies that the nornalizer of $H \cap K$ in G is equal to G. In other words, $H \cap K$ is a subgroup of order 16 that is normal in G. This shows that G is not simple.