(1) The average rate of growth is \((W(5) - W(4))/(5 - 4) = 0.1 \cdot 5^2 - 0.1 \cdot 4^2 = 2.5 - 1.6 = 0.9\) grams per week. The instantenous rate of growth is the value of the derivative \(W'(t) = 0.2t\) at \(t = 4\), so it is \(W'(4) = 0.8\) grams per week.

(2) The functions whose first derivative equal \(1/x^2\) are the antiderivatives \(-1/x + C_1\) where \(C_1\) is a constant. The antiderivatives of \(-1/x + C_1\) are \(-\ln(x) + C_1 x + C_2\) where \(C_2\) is another constant. Hence the functions whose second derivative equals \(1/x^2\) are precisely the functions \(f(x) = -\ln(x) + C_1 x + C_2\), where \(C_1\) and \(C_2\) are arbitrary constants.

(3) We use the method of implicit differentiation, where \(y = y(x)\) is regarded as a function of \(x\). Applying \(d/dx\) to both sides of the equation \(xy + y^3 = 14\), we find
\[
\frac{d}{dx}(xy + y^3) = y + x \frac{dy}{dx} + 3y^2 \frac{dy}{dx} = y + (x + 3y^2) \frac{dy}{dx} = 0,
\]
and hence \(dy/dx = -y/(x + 3y^2)\). Substituting \(x = 3, y = 2\) into this expression, we find that the slope of the tangent line equals \(-2/(3 + 3 \cdot 2^2) = -2/15\). The line with slope \(-2/15\) through \((3, 2)\) is found to be \(y = -(2/15)x + 12/5\), and this is our tangent line.

(4a) Using the Chain Rule, we find
\[
\frac{d}{dx}\ln(\ln(x)) = \frac{1}{\ln(x)} \cdot \frac{d}{dx}(\ln(x)) = \frac{1}{\ln(x)} \cdot \frac{1}{x} = \frac{1}{x \cdot \ln(x)}.
\]

(4b) Using Implicit Differentiation, we know that \(\frac{d}{dx} g(x)\) equals \(g(x)\) times
\[
\frac{d}{dx}\ln(g(x)) = \frac{2}{x + 1} + \frac{3}{x + 2} + \frac{5}{x + 3} + \frac{7}{x + 4} + \frac{11}{x + 5}.
\]
Therefore the derivative of \(g(x)\) equals
\[
((x + 1)^2(x + 2)^3(x + 3)^5(x + 4)^7(x + 5)^{11}) \cdot \left(\frac{2}{x + 1} + \frac{3}{x + 2} + \frac{5}{x + 3} + \frac{7}{x + 4} + \frac{11}{x + 5}\right).
\]

(5) The amount of radium after \(t\) years equals \(P(t) = 100 e^{-\lambda t}\) grams where \(\lambda\) is the decay constant. The average amount of radium over the next 1000 years equals
\[
\frac{1}{1000} \int_0^{1000} P(t)dt = \frac{1}{10} \int_0^{1000} e^{-\lambda t} dt = \frac{1}{10\lambda} \cdot (1 - e^{-\lambda 1000}).
\]
We know that the half life is 1600 years, so the decay constant is determined by the equation \(100e^{-\lambda \cdot 1600} = 50\), namely, we find \(\lambda = -\frac{1}{1600} \ln(1/2)\). Substituting this value of \(\lambda\) into the right hand side of (*) , we find that the average amount of radium over the next 1000 years equals \(-(1 - (1/2)5/8)/\ln(1/2)\) grams. We don’t simplify this number.
The conditions \( f' = f \) so the function we must maximize is \( f(G_F(2)) \) or by just trying, we find the function \( f(p) = 1/p^2 \) which solves this differential equation.

We use the Fundamental Theorem of Calculus to evaluate these definite integrals. An antiderivative of \( f(x) = 3/(x+2)^4 \) is \( F(x) = -(x+2)^{-3} \), so the first integral equals \( F(2) - F(-1) = -4^{-3} + 1^{-3} = 63/64 \). An antiderivative of \( g(x) = 7/e^{4x} = 7e^{-4x} \) is \( G(x) = -(7/4)e^{-4x} \), so the second integral equals \( G(1) - G(0) = (7/4)(1 - e^{-4}) \).

A solution to this differential equation is the function
\[
y = \frac{M}{1 + e^{-Mkt}}
\]
whose graph is the logistic growth curve. This curve and its differential equation are important in ecology, where it describes the growth of a population within limited resources. One example we discussed in class is the population of fish in a small lake. Here \( M \) is the maximum number of fish that the lake can support, and \( k \) is a parameter which determines the initial growth rate \( y'(0) = k \cdot y(0) \cdot (M - y(0)) = k \cdot M^2/4 \).

By solving the constraints for \( y \) and \( z \) we find \( y = 1 - x \) and \( z = 2 - y = 1 + x \), so the function we must maximize is \( f(x) = x(1 - x)(1 + x) = x - x^3 \). The derivative \( f'(x) = 1 - 3x^2 \) has its unique positive zero at \( x = 1/\sqrt{3} \), and this is a local maximum because \( f''(x) = -6x \) is negative at \( x = 1/\sqrt{3} \). Hence the maximal value of \( Q = xyz \) equals \( f(1/\sqrt{3}) = 2\sqrt{3}/9 \), and this is attained for \( x = 1/\sqrt{3}, y = 1 - 1/\sqrt{3}, z = 1 + 1/\sqrt{3} \).

We want \( f(n) = C_1a^n + C_2b^n \) for all \( n \geq 0 \). This implies the condition
\[
f(n) - f(n-1) - f(n-2) = C_1 \cdot a^{n-2} \cdot (a^2 - a - 1) + C_2 \cdot b^{n-2} \cdot (b^2 - b - 1) = 0.
\]
This suggests that we take \( a \) and \( b \) as the solutions of the equation \( x^2 - x - 1 = 0 \), namely,
\[
a = \frac{1 + \sqrt{5}}{2} \quad \text{and} \quad b = \frac{1 - \sqrt{5}}{2}.
\]
The conditions \( f(0) = 0 \) and \( f(1) = 1 \) imply \( C_1 + C_2 = 0 \) and \( C_1a + C_2b = 1 \), and therefore
\[
C_1 = \frac{1}{\sqrt{5}} \quad \text{and} \quad C_2 = -\frac{1}{\sqrt{5}}.
\]
We conclude that the \( n \)-th Fibonacci number is given by the explicit formula
\[
f(n) = \frac{1}{\sqrt{5}} \cdot \left( \frac{1 + \sqrt{5}}{2} \right)^n - \frac{1}{\sqrt{5}} \cdot \left( \frac{1 - \sqrt{5}}{2} \right)^n.
\]