Math 16A, Solutions to the Final Exam

(1) The average rate of growth is $(W(5)-W(4))/(5-4) = 0.1{\cdot}5^2 - 0.1{\cdot}4^2 = 2.5-1.6 = 0.9$ grams per week. The instantenous rate of growth is the value of the derivative $W'(t) = 0.2t$ at $t = 4$, so it is $W'(4) = 0.8$ grams per week.

(2) The functions whose first derivative equal $1/x^2$ are the antiderivatives $-1/x + C_1$ where C_1 is a constant. The antiderivatives of $-1/x+C_1$ are $-\ln(x)+C_1x+C_2$ where C_2 is another constant. Hence the functions whose second derivative equals $1/x^2$ are precisely the functions $f(x) = -\ln(x) + C_1x + C_2$, where C_1 and C_2 are arbitrary constants.

(3) We use the method of implicit differentiation, where $y = y(x)$ is regarded as a function of x. Applying d/dx to both sides of the equation $xy + y^3 = 14$, we find

$$\frac{d}{dx}(xy + y^3) \;=\; y + x\frac{dy}{dx} + 3y^2\frac{dy}{dx} \;=\; y + (x + 3y^2)\cdot\frac{dy}{dx} \;=\; 0,$$

and hence $dy/dx = -y/(x + 3y^2)$. Substituting $x = 3, y = 2$ into this expression, we find that the slope of the tangent line equals $-2/(3 + 3 \cdot 2^2) = -2/15$. The line with slope $-2/15$ through $(3, 2)$ is found to be $y = -(2/15)x + 12/5$, and this is our tangent line.

(4a) Using the Chain Rule, we find

$$\frac{d}{dx}\ln(\ln(x)) \;=\; \frac{1}{\ln(x)}\cdot\frac{d}{dx}(\ln(x)) \;=\; \frac{1}{\ln(x)}\cdot\frac{1}{x} \;=\; \frac{1}{x\cdot\ln(x)}.$$

(4b) Using Implicit Differentiation, we know that $\frac{d}{dx}g(x)$ equals $g(x)$ times

$$\frac{d}{dx}\ln(g(x)) \;=\; \frac{2}{x+1} + \frac{3}{x+2} + \frac{5}{x+3} + \frac{7}{x+4} + \frac{11}{x+5}.$$

Therefore the derivative of $g(x)$ equals

$$((x+1)^2(x+2)^3(x+3)^5(x+4)^7(x+5)^{11})\cdot\Big(\frac{2}{x{+}1} + \frac{3}{x{+}2} + \frac{5}{x{+}3} + \frac{7}{x{+}4} + \frac{11}{x{+}5}\Big).$$

(5) The amount of radium after t years equals $P(t) = 100e^{-\lambda t}$ grams where λ is the decay constant. The average amount of radium over the next 1000 years equals

$$\frac{1}{1000}\int_0^{1000} P(t)dt \;=\; \frac{1}{10}\int_0^{1000} e^{-\lambda t}dt \;=\; \frac{1}{10\lambda}\cdot(1 - e^{-\lambda 1000}). \qquad (*)$$

We know that the half life is 1600 years, so the decay constant is determined by the equation $100e^{-\lambda\cdot 1600} = 50$, namely, we find $\lambda = -\frac{1}{1600}\ln(1/2)$. Substituting this value of λ into the right hand side of $(*)$, we find that the average amount of radium over the next 1000 years equals $-(1 - (1/2)^{5/8})/\ln(1/2)$ grams. We don't simplify this number.

(6) The elasticity of demand for a demand function $f(p)$ equals

$$E(q) \;=\; -\frac{p\cdot f'(p)}{f(p)}.$$

In order for this expression to be the constant 2, the function $f(p)$ must satisfy the differential equation $p \cdot f'(p) = -2f(p)$. By remembering homework problem 24 on page 293, or by just trying, we find the function $f(p) = 1/p^2$ which solves this differential equation.

(7) We use the Fundamental Theorem of Calculus to evaluate these definite integrals. An antiderivative of $f(x) = 3/(x+2)^4$ is $F(x) = -(x+2)^{-3}$, so the first integral equals $F(2) - F(-1) = -4^{-3} + 1^{-3} = 63/64$. An antiderivative of $g(x) = 7/e^{4x} = 7e^{-4x}$ is $G(x) = -(7/4)e^{-4x}$, so the second integral equals $G(1) - G(0) = (7/4)(1 - e^{-4})$.

(8) A solution to this differential equation is the function

$$y \;=\; \frac{M}{1+e^{-Mkt}}$$

whose graph is the *logistic growth curve*. This curve and its differential equation are important in ecology, where it describes the growth of a population within limited resources. One example we discussed in class is the population of fish in a small lake. Here M is the maximum number of fish that the lake can support, and k is a parameter which determines the initial growth rate $y'(0) = k \cdot y(0) \cdot (M - y(0)) = k \cdot M^2/4$.

(9) By solving the constraints for y and z we find $y = 1 - x$ and $z = 2 - y = 1 + x$, so the function we must maximize is $f(x) = x(1-x)(1+x) = x - x^3$. The derivative $f'(x) = 1 - 3x^2$ has its unique positive zero at $x = 1/\sqrt{3}$, and this is a local maximum because $f''(x) = -6x$ is negative at $x = 1/\sqrt{3}$. Hence the maximal value of $Q = xyz$ equals $f(1/\sqrt{3}) = 2\sqrt{3}/9$, and this is attained for $x = 1/\sqrt{3}$, $y = 1 - 1/\sqrt{3}$, $z = 1 + 1/\sqrt{3}$.

(10) We want $f(n) = C_1a^n + C_2b^n$ for all $n \geq 0$. This implies the condition

$$f(n) - f(n-1) - f(n-2) \;=\; C_1 \cdot a^{n-2} \cdot (a^2 - a - 1) \;+\; C_2 \cdot b^{n-2} \cdot (b^2 - b - 1) \;=\; 0.$$

This suggests that we take a and b as the solutions of the equation $x^2 - x - 1 = 0$, namely,

$$a \;=\; \frac{1+\sqrt{5}}{2} \quad \text{and} \quad b \;=\; \frac{1-\sqrt{5}}{2}.$$

The conditions $f(0) = 0$ and $f(1) = 1$ imply $C_1 + C_2 = 0$ and $C_1a + C_2b = 1$, and therefore

$$C_1 \;=\; \frac{1}{\sqrt{5}} \quad \text{and} \quad C_2 = -\frac{1}{\sqrt{5}}.$$

We conclude that the n-th Fibonacci number is given by the explicit formula

$$f(n) \;=\; \frac{1}{\sqrt{5}} \cdot \Big(\frac{1+\sqrt{5}}{2}\Big)^n \;-\; \frac{1}{\sqrt{5}} \cdot \Big(\frac{1-\sqrt{5}}{2}\Big)^n.$$