Math 55, **Second Midterm Exam**
SOLUTIONS

(1) Part (a) is problem # 21 (d) on page 361. The desired permutations of the seven letters A, B, C, D, E, F and G are in bijection with the permutations of the four letters X, Y, F and G, where X represents the string ABC and Y represents the string DE. The number of permutations of any set with four elements is $4! = 24$. Hence the answer is **24**.

For part (b) we observe that there are $7! = 5040$ permutations in total, and in half of them the letter A precedes the letter B, and in the others the letter B precedes A. Switching A and B defines a bijection between these two subsets. The answer is **2520**.

(2) This is problem # 25 on page 440. First consider the case $n = 1$. If one fair coin is flipped, then $X_1(\text{heads}) = -1$, $X_1(\text{tails}) = 1$, the expected value is $E(X_1) = (1/2)(-1) + (1/2)(1) = 0$, and the variance is $V(X_1) = E(X_1^2) - E(X_1)^2 = 1 - 0 = 1$. By linearity of expectation (Theorem 3 on page 429), we have $E(X_n) = n \cdot E(X_1) = 0$. By linearity of variance for independent random variables (Theorem 7 on page 437), we have $V(X_n) = n \cdot V(X_1) = n$. Hence the random variable X_n has expected value **0** and variance **n**.

(3) This is (essentially) problem # 7 on page 217. We give a *proof by contradiction* that $\log_2(5)$ is not a rational number. Suppose, on the contrary, that $\log_2(5) = a/b$ for some relatively prime positive integers a and b. Then $5 = 2^{a/b}$ and therefore $5^b = 2^a$. By the Fundamental Theorem of Arithmetic (page 211), every integer has a *unique factorization into primes*. This shows that $5^b = 2^a$ is not possible unless $a = b = 0$. This is a contradiction to our assumption that a and b are positive, and the proof is complete.

(4) The recurrence relation $a_n = 6a_{n-1} - 9a_{n-2}$ has the characteristic equation $r^2 - 6r + 9 = (r-3)^2 = 0$. This equation has only one root, $r_0 = 3$, so we apply Theorem 2 on page 464 to find that the general solution to the recurrence is $a_n = \gamma_1 3^n + \gamma_2 n 3^n$ for some real numbers γ_1 and γ_2. The initial conditions tell us that $a_0 = \gamma_1 = 1$ and $a_1 = 3\gamma_1 + 3\gamma_2 = 9$, and this implies $\gamma_1 = 1$ and $\gamma_2 = 2$. Therefore the solution to the recurrence relation is

$$a_n \;=\; 3^n + 2 \cdot n \cdot 3^n \;=\; (1 + 2n) \cdot 3^n.$$

(5) The number of tosses follows a *geometric distribution* with parameter $p = 1/3$.
(a) The thumb tack must land point up four times in a row and then point down. The probability of this happening is $(1-p)^4 p = (2/3)^4(1/3) = \mathbf{16/243}$.
(b) The expected number of tosses in a geometric distribution is $1/p = \mathbf{3}$.