Math 55, **Second Midterm Exam**
SOLUTIONS

(1) This problem is # 9 in Section 6.2 on page 405 of Rosen's book. We apply the Generalized Pigeonhole Principle, where $k = 50$ is the number of states. We seek to find the smallest integer N such that $\lceil N/50 \rceil = 100$. That integer is $N = \mathbf{4951}$. Hence, 4951 students must enroll to guarantee that a set of 100 students comes from the same state.

(2) This problem is # 12 in the review of Chapter 1 on page 108 of Rosen's book. It is meant to illustrate the concept of an inconstructive proof, as we cannot evaluate those big integers and find their sign. The proof goes like this: If one of the three numbers is zero then the two products of that number with the other two numbers are both zero. If none of the three numbers is zero, then two of them must have the same sign. The product of two numbers that have the same nonzero sign is positive, and hence it is nonnegative.

(3) This problem is similar to # 3 and # 4 in Section 7.3 on page 475 of Rosen's book. We apply Bayes' Theorem. Let E denote the event that Mike has chosen a red ball, and let F denote the event that Mike chose from the second box. Since Mike picks the box at random, we have $p(F) = p(\overline{F}) = 1/2$. Similarly, from the numbers of balls in the problem description, we know that $p(E|F) = 2/7$ and $p(E|\overline{F}) = 4/7$. Hence the answer is

$$p(F|E) \;=\; \frac{p(E|F)p(F)}{p(E|F)p(F) + p(E|\overline{F})p(\overline{F})} \;=\; \frac{p(E|F)}{p(E|F) + p(E|\overline{F})} \;=\; \frac{2/7}{2/7 + 4/7} \;=\; \mathbf{\frac{1}{3}}.$$

(4) The number of dice rolls follows a *geometric distribution* with parameter $p = 1/6$.

(a) The die must land on a number ≤ 5 three times in a row and thereafter it lands on the 6. The probability of this happening is $(1-p)^3 p = (5/6)^3(1/6) = \mathbf{125/1296}$.

(b) The expected number of rolls in a geometric distribution is $1/p = \mathbf{6}$.

(5) The two problems are solved using the methods explained in Section 6.5.

(a) This is the same as counting the number of 8-combinations for a set with 5 elements when repetitions are allowed. That number is

$$\binom{8+5-1}{8} \;=\; \binom{12}{8} \;=\; \binom{12}{4} \;=\; \frac{12 \cdot 11 \cdot 10 \cdot 9}{1 \cdot 2 \cdot 3 \cdot 4} \;=\; \mathbf{495}.$$

(b) This is the number of partitions of the integer 8 into at most 5 parts. The are precisely **18** such partitions: [1+1+2+2+2], [2+2+2+2], [1+1+1+2+3], [1+2+2+3], [1+1+3+3], [2+3+3], [1+1+1+1+4], [1+1+2+4], [2+2+4], [1+3+4], [4+4], [1+1+1+5], [1+2+5], [3+5], [1+1+6], [2+6], [1+7], [8].