Math 55, **First Midterm Exam**
SOLUTIONS

(1) There are twelve positive integers less than 36 that are relatively prime to 36. Hence $\phi(36) = \mathbf{12}$. Since 37 is a prime number, we have $\phi(37) = 37 - 1 = \mathbf{36}$. In the second question of Quiz # 2 we saw that $\phi(p^k) = p^k - p^{k-1}$ for any power of a prime number p. Therefore, $\phi(81) = \phi(3^4) = 3^4 - 3^3 = \mathbf{54}$ and $\phi(1024) = \phi(2^{10}) = 2^{10} - 2^9 = \mathbf{512}$.

(2) It is important to note that the domain is the set of nonnegative integers.

(a) The statement $\exists x \, ((x^2 < 10) \wedge (|3 - x| > 2))$ is **true** because $x = 0$ satisfies both inequalities in the conjunction.

(b) The statement $\forall x \, ((x \neq 4) \rightarrow (x - 5 > 1))$ is **false**. It does not hold for $x = 3$.

(c) The statement $\forall x \, \exists y \, (x + y = 0)$ is **false** because positive integers have no additive inverses among the nonnegative integers.

(d) The statement $\exists x \, \forall y \, (xy = 0)$ is **true** because $x = 0$ is a nonnegative integer, and it satisfies $0 \cdot y = 0$ for all y.

(3) We use induction on n. The induction base is $n = 1$. Here the statement is true because 5 divides $1^5 - 1 = 0$. For the induction step, we now assume that the statement is true for $n = k$ and we shall prove that it also holds for $n = k + 1$. Consider the identity

$$(k+1)^5 - (k+1) \quad = \quad 5 \cdot (k^4 + 2k^3 + 2k^2 + k) \; + \; (k^5 - k).$$

According to the induction hypothesis, the expression in the second parenthesis is divisible by 5. Since the expression in the first parenthesis is an integer, we conclude that the left hand side is divisible by 5. Hence 5 divides $n^5 - n$ for all positive integers n.

(4) We use the Extended Euclidean Algorithm to check that $gcd(81, 250) = 1$.

$$\begin{array}{ccc} 250 & = & 3 \cdot 81 + 7 \\ 81 & = & 11 \cdot 7 + 4 \\ 7 & = & 1 \cdot 4 + 3 \\ 4 & = & 1 \cdot 3 + 1 \end{array}$$

By back substitution we find that

$$1 = 2 \cdot 4 - 7 = 2 \cdot (81 - 11 \cdot 7) - 7 = 2 \cdot 81 - 23 \cdot 7 = 2 \cdot 81 - 23 \cdot (250 - 3 \cdot 81) = \mathbf{71} \cdot 81 - 23 \cdot 250.$$

Hence 71 is an inverse to 81 modulo 250.

(5) **Yes**, the symmetric difference operation on sets is **associative**. By trying out all eight possibilities in a *membership table*, we find that $A \oplus (B \oplus C) = (A \oplus B) \oplus C$, and this

set consist of all elements that lie either in all three sets or in precisely one of them:

A	B	C	$A \oplus B$	$B \oplus C$	$(A \oplus B) \oplus C$	$A \oplus (B \oplus C)$
T	T	T	F	F	T	T
T	T	F	F	T	F	F
T	F	T	T	T	F	F
T	F	F	T	F	T	T
F	T	T	T	F	F	F
F	T	F	T	T	T	T
F	F	T	F	T	T	T
F	F	F	F	F	F	F

Here "T" represents the statement that an arbitrary element of the domain is contained in the set labeling that column, and "F" represents the statement that the element not is contained in the set labeling that column.