(1) This problem is # 31 in Section 1.5 on page 67 of Rosen’s book. We use the methods explained in the subsection Negating Nested Quantifiers on pages 63-64 to solve these.

(a) The negation is obtained by switching the quantifiers: \( \exists x \forall y \exists z \neg T(x, y, z) \).

(b) Negation takes disjunction into conjunction: \( \exists x \forall y \neg P(x, y) \land \exists x \forall y \neg Q(x, y) \).

(c) Conjunction becomes disjunction: \( \exists x \forall y (\neg P(x, y) \lor \forall z \neg R(x, y, z)) \).

(d) We use that \( p \rightarrow q \) means \( \neg p \lor q \) to find the negation \( \exists x \forall y (P(x, y) \land \neg Q(x, y)) \).

(2) We use the method described in the proof of the Chinese Remainder Theorem. We set \( m_1 = 7, m_2 = 8, m_3 = 9 \), as well as \( M_1 = 8 \cdot 9 = 72, M_2 = 7 \cdot 9 = 63 \) and \( M_3 = 7 \cdot 8 = 56 \). For \( i = 1, 2, 3 \) we need to compute an inverse \( y_i \) of \( M_i \) modulo \( m_i \).

We can take \( y_1 = 4 \) because \( 4 \cdot 72 \equiv 4 \cdot 2 \equiv 1 \pmod{7} \), we can take \( y_2 = -1 \) because \((-1) \cdot 63 \equiv (-1) \cdot (-1) \equiv 1 \pmod{8} \), and we can take \( y_3 = 5 \) because \( 5 \cdot 56 \equiv 5 \cdot 2 \equiv 1 \pmod{9} \).

The desired solution is found to be

\[
1y_1M_1 + 3y_2M_2 + 2y_3M_3 = 1 \cdot 4 \cdot 72 + 3 \cdot (-1) \cdot 63 + 2 \cdot 5 \cdot 56 = 288 - 189 + 560 = 659.
\]

If we subtract \( 504 = 7 \cdot 8 \cdot 9 \) then we obtain the smallest positive integer solution \( n = 155 \).

(3) The largest amount of postage which cannot be formed with 5-cent and 6-cent stamps is 29 cents. This can be found by trial and error, or by the computing Frobenius number as \( 5 \cdot 6 - 5 - 6 = 19 \). Below 19 cents, precisely the following positive integers can be expressed:

\[
5, 6, 10, 11, 12, 15, 16, 17, 18.
\]

Our conjecture states: Every amount of postage of 20 cents or more can be formed using just 5-cent and 6-cent stamps. The proof is analogous to that in Example 4 on page 337:

We will use strong induction to prove this result. Let \( P(n) \) be the statement that postage of \( n \) cents can be formed using 5-cent and 6-cent stamps.

\textbf{Basis step}: The propositions \( P(20), P(21), P(22), P(23) \) and \( P(24) \) are true because \( 20 = 5+5+5+5, 21 = 5+5+5+6, 22 = 5+5+6+6, 23 = 5+6+6+6, \) and \( 24 = 6+6+6+6 \).

\textbf{Inductive step}: The inductive hypothesis is the statement \( P(j) \) is true for \( 20 \leq j \leq k \) where \( k \) is an integer with \( k \geq 24 \). Assuming this to be the case, we need to show that \( P(k+1) \) is true. Using the inductive hypothesis we can assume that \( P(k-4) \) is true because \( k-4 \) must be at least 20. Hence we can form postage of \( k-4 \) cents using 5-cent and 6-cent stamps. Just add one extra 5-cent stamp to the solution for \( k-4 \) cents and get postage for \( k+1 \) cents. Hence \( P(k+1) \) is true.

Because we have completed the basis step and the inductive step of a strong induction proof, we know by strong induction that \( P(n) \) is true for all integers \( n \geq 20 \).
(4) We use Fermat’s Little Theorem (Theorem 3 in Section 4.4 on page 281) which states that \( a^{p-1} \equiv 1 \pmod{p} \) for all primes \( p \) and all integers \( a \) not divisible by \( p \).

(a) By Fermat’s Little Theorem, we have \( 19^{12} \equiv 1 \pmod{13} \), and therefore
\[
19^{145} = 19^{12 \cdot 12 + 1} = (19^{12})^{12} \cdot 19 \equiv 1^{12} \cdot 19 = 19 \equiv 6 \pmod{13}.
\]

(b) By Fermat’s Little Theorem, we have \((-12)^6 \equiv 1 \pmod{7}\). We also note that \(50 \equiv 1 \pmod{7}\). We conclude \((-12)^{36} \cdot 50^{19} \equiv ((-12)^6)^{6} \cdot 1^{19} \equiv 1 \pmod{7}\).

(5) This problem is #11 in Section 2.5 on page 176 of Rosen’s book. The set \( \mathbb{Z}^+ \) of all positive integers is countably infinite. We also use the fact that any closed interval \([a, b]\), where \( a \) and \( b \) are real numbers satisfying \( a < b \), is an uncountable set.

(a) Let \( A = [0, 1] \) and \( B = [1, 2] \). Then \( A \cap B = \{1\} \), which is a finite set.

(b) Let \( A = [0, 1] \cup \mathbb{Z}^+ \) and \( B = [1, 2] \cup \mathbb{Z}^+ \). Then \( A \cap B = \mathbb{Z}^+ \) is countably infinite.

(c) Let \( A = [0, 1] \) and \( B = [0, 1] \). Then \( A \cap B = [0, 1] \) is uncountable.