Math 275: Geometry of Convex Optimization

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Homework # 2, due Tuesday, September 21

1. Please tell me a little bit about yourself and your interests. What is your mathematical background, what are you generally working on, who is your advisor/mentor, and what do you hope to get out of this class. Two paragaphs are definitely enough, either e-mail or hard copy.

2. A *correlation matrix* is a positive semidefinite real symmetric $n \times n$-matrix all of whose diagonal entries are 1.

   (a) Maximize the sum of the off-diagonal entries over correlation matrices with $n = 3$. Solve this optimization problem also for $n = 4$.

   (b) Minimize the sum of the off-diagonal entries over correlation matrices with $n = 3$. Solve this optimization problem also for $n = 4$.

   (c) Does there exist a correlation matrix, of any size n, whose determinant is larger than 1? Find a proof or counterexample.

3. Consider the *Birkhoff polytope* all non-negative 4×4-matrices whose rows and columns sum to 1. Determine the f-vector of this polytope.

4. Let P be the 4-dimensional *transportation polytope* consisting of all non-negative 3×3-matrices with row sums $4, 6, 7$ and column sums $3, 5, 9$.

   (a) Maximize the trace over all 3×3-matrices in P.

   (b) Determine the *analytic center* of P, i.e., maximize the product of the nine matrix entries over all 3×3-matrices in P.

   (c) Compute the *central path* for the linear program in part (a).

5. Draw a *Schlegel diagram* of the polytope P in the previous problem.

6. Consider the curve in 4-dimensional space with parametrization

$$t \mapsto \big(\cos(t), \cos(2t), \cos(3t), \cos(4t)\big).$$

   Show that this is an algebraic curve, and determine its prime ideal. Show that the convex hull of this curve is a spectrahedron. Describe all faces of that spectrahedron. What is the optimal value function?

7. Consider the ideal $I = \langle x^2 + y^2 - 1 \,,\, xy - 3 \rangle$ in $\mathbb{Q}[x, y]$.

   (a) Determine the variety $\mathcal{V}_{\mathbb{C}}(I)$ and show that it has no real points.

   (b) According to the *Real Nullstellensatz*, the polynomial -1 can be written as a sum of squares in $\mathbb{Q}[x, y]/I$. Find such an expression.

   (c) Compute the *real radical* of I using the semdefinite programming algorithm of Laurent-Lasserre-Rostalski in `arXiv:math/0609528`.