Math 275: Geometry of Convex Optimization

Bernd Sturmfels, UC Berkeley, Fall 2010
Homework # 1, due Tuesday, September 7

1. Determine the global minimum in $\mathbb{R}^2$ of the polynomial

$$f(x,y) \quad = \quad x^6 + y^6 + x^3 + y^3 + xy.$$

   Express the optimal points and the optimal value in radicals over $\mathbb{Q}$.

2. Let $f(x,y)$ be the polynomial in the previous problem and consider the problem of maximizing λ such that $f(x,y) - \lambda$ is a sum of squares.
   - Write this as a semidefinite program (SDP) for 10×10-matrices.
   - Solve the SDP numerically, and compare with your answer in #1.
   - The given polynomial is symmetric in x and y. Can you formulate an equivalent SDP that takes advantage of this symmetry?

3. Determine the location in California that minimizes the sum of the distances to the five largest cities: *Los Angeles*, *San Diego*, *San Jose*, *San Francisco* and *Fresno*. For coordinates see `www.travelmath.com`.

4. The curve $X = \{(x,y) \in \mathbb{R}^2 \,:\, x^4 + y^4 = 1\}$ is known as the *TV screen*.
   - Is the convex hull of the TV screen X a spectrahedron?
   - Find the irreducible polynomial whose zero set is the dual curve X^*.
   - Can you write the equation $x^4 + y^4 - 1$ of X as the determinant of a symmetric 4×4-matrix over $\mathbb{C}$ whose entries are linear in x, y?

5. The polynomial $p(x) = 1 + x + x^2 + x^3 + x^4 + x^5 + x^6$ is non-negative on the real line. What is its minimum value? Write $p(x)$ as a sum of squares. The set of all sums of squares representations of $p(x)$ is a 3-dimensional spectrahedron. Draw a picture of this spectrahedron. Determine all possible representations of $p(x)$ as a sum of <u>two</u> squares.

6. Intersect the unit sphere in 3-space with a general quadratic surface. Show that the convex hull of the resulting curve C is a spectrahedron.

7. Let P be the convex hull of the union of two circles in 3-dimensional space, where the first circle is defined by $x^2 + y^2 = 5/4$ and $z = 0$, and the second circle is defined by $x^2 + z^2 = 1$ and $y = 0$. Compute the irreducible polynomial in x, y, z that vanishes on the boundary of P. You can find pictures at the home page of *Tina Mai* at Texas A&M.

8. Let P be the convex hull of a compact curve C of degree 6 in $\mathbb{R}^2$.

   - What is the expected degree of the algebraic boundary of the convex set $P = \mathrm{conv}(C)$, assuming the curve C is chosen at random?
   - What is the maximum number of ovals a sextic curve C can have?
   - Construct a curve C where P has as many straight edges as possible. The submission with the most edges wins a **candy prize**.