Math 275: Introduction to Non-Linear Algebra

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Homework # 7, due Monday, March 10

1. The symmetry group of a regular square in $\mathbb{R}^2$ acts naturally on $\mathbb{R}[x, y]$. Determine the subring of invariants. Start by guessing some invariants in small degree, and check completeness using the Molien series.

2. Prove *Noether's degree bound*: If G is a finite subgroup of $\mathrm{GL}(n, \mathbb{C})$ then the invariant ring $\mathbb{C}[x_1, \ldots, x_n]^G$ is generated as a $\mathbb{C}$-algebra by polynomials of degree at most the group order $|G|$.

3. The multiplicative group $G = \mathbb{C}^*$ acts on the polynomial ring $\mathbb{C}[a, b, c, d]$ by sending a to ta, b to $t^3 b$, c to c/t^2, and d to d/t^2. Determine a finite generating set for the invariant ring $\mathbb{C}[a, b, c, d]^G$.

4. Consider the action of $SL_2(\mathbb{C})$ by simultaneous conjugation on the space of pairs of 2×2-matrices. What is the ring of invariants?

5. The action of $SL_3(\mathbb{C})$ on the space $V = S^4\mathbb{C}^3$ of ternary quartics has an invariant of degree 3. Write this invariant explicitly as a polynomial in 15 unknowns. Can you guess what its geometric meaning might be?

6. Give an example of a real $2 \times 2 \times 2$-tensor whose tensor rank over $\mathbb{C}$ is 2 but whose tensor rank over $\mathbb{R}$ is 3. (Hint: [De Silva & Lim, 2008]).

7. The matrix $M = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$ has rank 3. Prove that the non-negative rank of M is equal to 4; i.e. show that M cannot be written as the product of a non-negative 4×3-matrix and a non-negative 3×4-matrix.