Math 275: Introduction to Non-Linear Algebra

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Homework # 6, due Monday, March 3

1. Minimize the trace over all positive semidefinite 3×3-matrices whose off-diagonal entries are $5, 6$ and 7. Express your answer in radicals.

2. The following simple polynomial system has no real solutions:
$$x^2 + y^2 \leq 1 \quad \text{and} \quad xy > 5.$$
Find a certificate as described in the Real Nullstellensatz.

3. Let I be the ideal in $\mathbb{R}[x, y, z]$ generated by the polynomials
$$x^2 + y^2 + z^2 - 1,\ xyz - 5 \ \text{ and } \ x + 2y + 3z - 7.$$
Write -1 as a sum of squares modulo I.

4. Draw the spectrahedron $\left\{ (x, y, z) \in \mathbb{R}^3 : \begin{bmatrix} 1 & x & y & z \\ x & 1 & z & y \\ y & z & 1 & x \\ z & y & x & 1 \end{bmatrix} \succeq 0 \right\}$.

5. Show that the set of non-negative degree 6 polynomials
$$ax^6 + bx^5 + cx^4 + dx^3 + ex^2 + fx + g$$
is a full-dimensional convex cone in the coefficient space $\mathbb{R}^7$. Find an explicit polynomial that vanishes on the boundary of this cone.

6. Consider the following polynomial of degree 6 in two variables:
$$f(x, y) \quad = \quad x^4y^2 + x^2y^4 - 3x^2y^2 + 1.$$
Prove that $f(x, y)$ is non-negative on $\mathbb{R}^2$ but it is not a sum of squares.

7. True or false: If $g(x, y)$ is a polynomial of degree 4 that is nonnegative at all points in $\mathbb{R}^2$ then $g(x, y)$ can be written as a sum of squares.