Critical Equations

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We consider a model $X_{\mathbb{R}}$ that is given as the zero set in $\mathbb{R}^n$ of a collection $\{f_1, \ldots, f_k\}$ of nonlinear polynomials in n unknowns $x_1, \ldots, x_n$. Thus, $X_{\mathbb{R}}$ is a real algebraic variety. In order to apply algebraic methods, we prefer to work with the complex algebraic variety $X \subset \mathbb{C}^n$ defined by the same polynomials. Thus $X_{\mathbb{R}}$ is the subset of real points in X.

We assume that X is irreducible, that $I_X = \langle f_1, \ldots, f_k \rangle$ is its prime ideal, and that the set of nonsingular real points is Zariski dense in X. The $k \times n$ Jacobian matrix $\mathcal{J} = (\partial f_i / \partial x_j)$ has rank at most c at any point $x \in X$, where $c = \operatorname{codim}(X)$. The point x is *nonsingular* on X if the rank is exactly c. The variety X is called *smooth* if all its points are nonsingular. Elaborations on these hypotheses are found in many text books, including [8, Chapter 2].

The following optimization problem arises in many applications. Given a data point $u \in \mathbb{R}^n \backslash X$, compute the distance to the model X. Thus, we seek a point x^* in X that is closest to u. The answer depends on the chosen metric. We focus on the case when the metric is represented by a polynomial and x^* is a smooth point on X. The optimal point x^* is a solution to the *critical equations*. In optimization, these are also known as first-order conditions or KKT equations, and they arise from introducing Lagrange multipliers. We seek to compute all complex solutions to the critical equations. The set of these *critical points* is typically finite, and it includes all local maxima, all local minima and all saddle points.

We begin by discussing the *Euclidean distance (ED) problem*, which is as follows:

$$\text{minimize } \sum_{i=1}^{n} (x_i - u_i)^2 \text{ subject to } x \in X. \tag{1}$$

Our first step is to derive the critical equations for (1). The *augmented Jacobian matrix* $\mathcal{AJ}$ is the $(k+1) \times n$ matrix obtained by placing the row $(x_1 - u_1, \ldots, x_n - u_n)$ atop the Jacobian matrix $\mathcal{J}$. We form the ideal generated by its $(c+1) \times (c+1)$ minors, we add the ideal of the model I_X, and we then saturate that sum by the ideal of $c \times c$ minors of $\mathcal{J}$. See [4, Eqn. (2.1)]. The result is the *critical ideal* $\mathcal{C}_{X,u}$ of the model X with respect to the data u.

Example 1 (Plane curves)**.** Let X be the plane curve defined by a polynomial $f(x_1, x_2)$. We wish to compute the Euclidean distance from X to a given point $u = (u_1, u_2) \in \mathbb{R}^2$. To this end, we form the augmented Jabobian matrix. This matrix is square of size 2×2:

$$\mathcal{AJ} \;=\; \begin{pmatrix} x_1 - u_1 & x_2 - u_2 \\ \partial f / \partial x_1 & \partial f / \partial x_2 \end{pmatrix} \tag{2}$$

The critical ideal is obtained from f and the determinant of $\mathcal{AJ}$ as follows:

$$\mathcal{C}_{X,u} \;=\; \langle\, f,\, \det(\mathcal{AJ})\,\rangle \;:\; \langle\, \partial f/\partial x_1, \partial f/\partial x_2 \,\rangle^{\infty}. \tag{3}$$

This ideal lives in $\mathbb{R}[x_1, x_2]$. Frequently, the coefficients of f and the coordinates of u are rational numbers, and in this case we can perform the computation purely symbolically in $\mathbb{Q}[x_1, x_2]$. The saturation step in (3) removes points that are singular on the curve $X = \mathcal{V}(f)$. If X is smooth then saturation is unnecessary, and we simply have $\mathcal{C}_{X,u} = \langle\, f,\, \det(\mathcal{AJ})\,\rangle$.

In applications, we must expect singularities. For a concrete example take the cardioid

$$f \;=\; (x_1^2 + x_2^2 + x_2)^2 - (x_1^2 + x_2^2), \tag{4}$$

and fix a random point $u = (u_1, u_2)$. See [4, Example 1.1]. The ideal $\langle\, f,\, \det(\mathcal{AJ})\,\rangle$ is the intersection of $\mathcal{C}_{X,u}$ and an $\langle x_1, x_2\rangle$-primary ideal of multiplicity 3. The critical ideal $\mathcal{C}_{X,u}$ has three distinct complex zeros. We can express their coordinates in radicals in u_1, u_2. $\diamond$

The variety $\mathcal{V}(\mathcal{C}_{X,u})$ is the set of complex critical points of (1). For random data u, this variety is a finite subset of $\mathbb{C}^n$, and it contains the optimal solution x^*, provided the latter is attained at a smooth point of X. It was proved in [4] that the number of critical points, i.e. the cardinality of the variety $\mathcal{V}(\mathcal{C}_{X,u})$, is independent of u, if we assume that the data point u is sufficiently general. This number is called the *ED degree* of the variety X. In Example 1 we examined a plane curve of degree 4 whose ED degree equals 3. The ED degree of a variety X measures the difficulty of solving the ED problem (1) using exact algebraic methods. The ED degree is an important complexity measure in metric algebraic geometry.

Example 2 (Space curves)**.** Fix $n = 3$ and let X be the curve in $\mathbb{R}^3$ defined by two general polynomials f_1 and f_2 of degrees d_1 and d_2 in x_1, x_2, x_3. The augmented Jacobian matrix is

$$\mathcal{AJ} \;=\; \begin{pmatrix} x_1 - u_1 & x_2 - u_2 & x_3 - u_3 \\ \partial f_1/\partial x_1 & \partial f_1/\partial x_2 & \partial f_1/\partial x_3 \\ \partial f_2/\partial x_1 & \partial f_2/\partial x_2 & \partial f_2/\partial x_3 \end{pmatrix}. \tag{5}$$

Fix a general data vector $u \in \mathbb{R}^3$. Then the critical ideal equals $\mathcal{C}_{X,u} = \big\langle f_1,\, f_2,\, \det(\mathcal{AJ})\big\rangle$. Hence the set of critical points is the intersection of three surfaces, of degrees d_1, d_2 and $d_1 + d_2 - 1$. By Bézout's Theorem [8, Theorem 2.16], the expected number of solutions is the product of these degrees. Hence the ED degree of the curve X equals $d_1 d_2 (d_1 + d_2 - 1)$.

The same formula can be derived from a formula for general curves in terms of algebraic geometry data. Let X be a general smooth curve of degree d and genus g in any ambient space $\mathbb{R}^n$. By [4, Corollary 5.9], we have $\mathrm{EDdegree}(X) = 3d + 2g - 2$. The above curve in 3-space has degree $d = d_1 d_2$ and genus $g = d_1^2 d_2/2 + d_1 d_2^2/2 - 2d_1 d_2 + 1$. We conclude that

$$\mathrm{EDdegree}(X) \;=\; 3d + 2g - 2 \;=\; d_1 d_2 (d_1 + d_2 - 1).$$

This formula also covers the case of plane curves (cf. Example 1). Namely, if we set $d_1 = d$ and $d_2 = 1$ then we see that a general plane curve X of degree d has $\mathrm{EDdegree}(X) = d^2$. In particular, a general plane quartic has ED degree 16. However, that number can drop a lot for curves that are special. For the cardiod in (4) the ED degree drops from 16 to 3. $\diamond$

Here is a general upper bound on the ED degree in terms of the given polynomials.

Proposition 3. *Let X be a variety of codimension c in $\mathbb{R}^n$ whose ideal I_X is generated by polynomials $f_1, f_2, \ldots, f_c, \ldots, f_k$ of degrees $d_1 \geq d_2 \geq \cdots \geq d_c \geq \cdots \geq d_k$. Then*

$$\textit{EDdegree}(X) \quad \leq \quad d_1 d_2 \cdots d_c \cdot \sum_{i_1+i_2+\cdots+i_c \leq n-c} (d_1-1)^{i_1}(d_2-1)^{i_2}\cdots(d_c-1)^{i_c}. \tag{6}$$

Equality holds when X is a generic complete intersection of codimension c (hence $c = k$).

Proof. This appears in [4, Proposition 2.6]. We can derive it as follows. Bézout's Theorem ensures that the degree of the variety X is at most $d_1 d_2 \cdots d_c$. The entries in the ith row of the matrix $\mathcal{AJ}$ are polynomials of degrees $d_i - 1$. The degree of the variety of $(c+1)\times(c+1)$ minors of $\mathcal{AJ}$ is at most the sum in (6). This follows from the Giambelli–Thom–Porteous formula, which expresses the degree of a determinantal variety in terms of symmetric functions. The intersection of that determinantal variety with X is our set of critical points, and the cardinality of that set is bounded by the product of the two degrees. Generically, that intersection is a complete intersection and the inequality (6) is attained. □

Formulas or a priori bounds for the ED degree are important when studying exact solutions to the optimization problem (1). The paradigm is to compute all complex critical points, by either symbolic or numerical methods, and to then extract one's favorite real solutions among these. This leads, for instance, to all local minima in (1). The ED degree is an upper bound on the number of real critical points, but this bound is generally not tight.

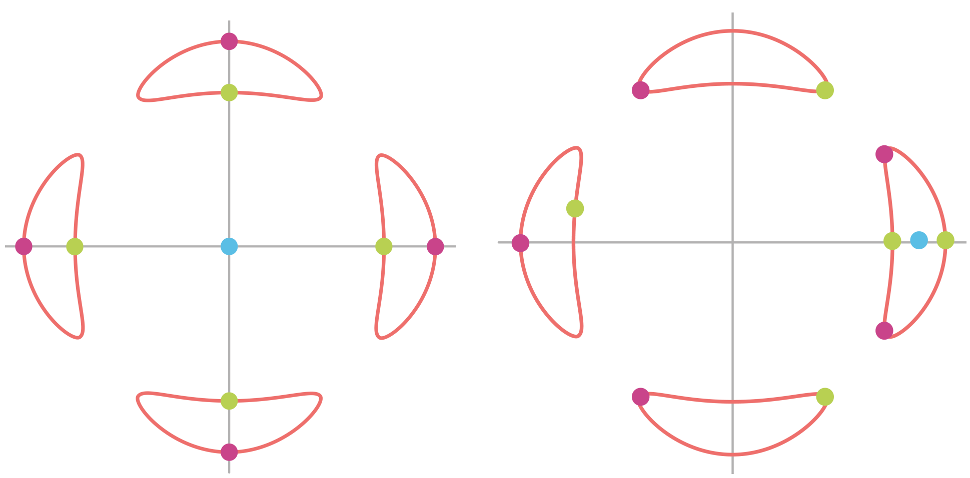

Figure 1: ED problems on the Trott curve: configurations of eight (left) or ten (right) critical points. Data points are blue, local minimal are green, and local maxima are purple. The coordinates of the critical points are computed by solving the critical equations in (7).

Example 4. Consider the case $n = 2, c = 1, d_1 = 4$ in Proposition 3, where X is a generic quartic curve in the plane $\mathbb{R}^2$. The number of complex critical points is $\mathrm{EDdegree}(X) = 16$. But, they cannot be all real. For an illustration, consider the *Trott curve* $X = V(f)$, given by

$$f \;=\; 144(x_1^4 + x_2^4) \;-\; 225(x_1^2 + x_2^2) \;+\; 350x_1^2x_2^2 \;+\; 81.$$

This curve is shown in Figure 1. For general data $u = (u_1, u_2)$ in $\mathbb{R}^2$, the critical equations

$$f \;=\; \frac{\partial f}{\partial x_2}(x_1 - u_1) - \frac{\partial f}{\partial x_1}(x_2 - u_2) \;=\; 0. \tag{7}$$

have distinct 16 complex solutions, and these are all critical points in X. Since the Trott curve is smooth, the saturation step in (3) is not needed when computing the ideal $\mathcal{C}_{X,u}$.

The ED degree 16 is an upper bound for the number of real critical points of the optimization problem (1) for any data point u. The actual number depends heavily on the specific location of u. For u near the origin, eight of the 16 points in $\mathcal{V}(\mathcal{C}_{X,u})$ are real. For $u = \left(\frac{7}{8}, \frac{1}{100}\right)$, which is inside the rightmost oval, there are 10 real critical points. The two scenarios are shown in Figure 1. Local minima are green, while local maxima are purple. For $u = (2, \frac{1}{100})$, to the right of the rightmost oval, the number of real critical points is 12. $\diamond$

In general, our task is to compute the complex zeros of the critical ideal $C_{X,u}$. Algorithms for this computation can be either symbolic or numerical. Symbolic methods usually rest on the construction of a Gröbner basis, to be followed by a floating point computation to extract the solutions. In recent years, numerical methods have become increasingly popular. These are based on homotopy continuation. Two notable packages are `Bertini` [1] and `HomotopyContinuation.jl` [3]. The ED degree is important here because it indicates how many paths need to be tracked to solve (1). We next illustrate current capabilities.

Example 5. Suppose X is defined by $c = k = 3$ random polynomials in $n = 7$ variables, for a range of degrees d_1, d_2, d_3. The table below lists the ED degree in each case, and the times used by `HomotopyContinuation.jl` to compute and certify all critical points in $\mathbb{C}^7$.

$d_1\, d_2\, d_3$	3 2 2	3 3 2	3 3 3	4 2 2	4 3 2	4 3 3	4 4 2	4 4 3
EDdegree	1188	3618	9477	4176	10152	23220	23392	49872
Solving (sec)	3.849	21.06	61.51	31.51	103.5	280.0	351.5	859.3
Certifying (sec)	0.390	1.549	4.653	2.762	7.591	17.16	21.65	50.07

Here we represent $C_{X,u}$ by a system of 10 equations in 10 variables. In addition to the three equations $f_1 = f_2 = f_3 = 0$ in $x_1, \ldots, x_7$, we take the seven equations $(1, y_1, y_2, y_3) \cdot \mathcal{AJ} = 0$. Here y_1, y_2, y_3 are new variables. These ensure that the 4×7 matrix $\mathcal{AJ}$ has rank ≤ 3. This formulation avoids the listing of all $\binom{7}{4} = 35$ maximal minors. It is the preferred representation of determinantal varieties in the setting of numerical algebraic geometry.

The timings above refer to computing all complex solutions to the system of 10 equations in 10 variables. They include the certification step [2] that proves correctness and completeness. These computations were performed using `HomotopyContinuation.jl` v2.5.6 on a 16 GB MacBook Pro with an Intel Core i7 processor working at 2.6 GHz. They suggest that our critical equations can be solved fast and reliably, with proof of correctness, when the ED degree is less than 50000. When the ED degree exceeds 50000, success with numerical path tracking will depend on the specific structure of the family. A key player on the geometric side is the discriminant of the problem. If that is well-behaved, then even larger ED degrees are feasible. A successful application to a physics problem is reported in [11, Table 1]. $\diamond$

When the ED problem (1) arises in an application then the variety X often describes matrices of low rank that are constrained to have a special structure. Sometimes these matrices are flattenings of tensors. This version of the problem was studied in the article [9], which focuses on Hankel matrices, Sylvester matrices and generic subspaces of matrices, and which uses a weighted version of the Euclidean metric. We now offer a brief introduction.

Our point of departure is the following general low-rank approximation problem:

$$\text{minimize} \;\; ||\, X - U \,||_\Lambda^2 \;=\; \sum_{i=1}^{m}\sum_{j=1}^{n} \lambda_{ij}(x_{ij}-u_{ij})^2 \quad \text{subject to} \quad \operatorname{rank}(X) \leq r. \tag{8}$$

Here, we are given a real *data matrix* $U = (u_{ij})$ of format $m \times n$, and we wish to find a matrix $X = (x_{ij})$ of rank at most r that is closest to U in a weighted Frobenius norm. The entries of the *weight matrix* $\Lambda = (\lambda_{ij})$ are positive reals. If $m \leq n$ and the weight matrix Λ is the all-one matrix $\mathbf{1}$ then the solution to (8) is given by the *singular value decomposition*

$$U \;=\; T_1 \cdot \operatorname{diag}(\sigma_1, \sigma_2, \ldots, \sigma_m) \cdot T_2.$$

Here T_1, T_2 are orthogonal matrices, and $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_m$ are the singular values of U. The following well-known theorem concerns the variety X of $m \times n$ matrices of rank $\leq r$.

Theorem 6 (Eckart-Young)**.** *The closest matrix of rank $\leq r$ to the given matrix U equals*

$$U^* \;=\; T_1 \cdot \operatorname{diag}(\sigma_1, \ldots, \sigma_r, 0, \ldots, 0) \cdot T_2. \tag{9}$$

This is the unique local minimum. All complex critical points are real. They are found by substituting zeros for $m-r$ of the entries of $\operatorname{diag}(\sigma_1, \ldots, \sigma_m)$*. Hence,* $\operatorname{EDdegree}(X) = \binom{m}{r}$.

For general weights Λ, the situation is more complicated. In particular, there can be complex critical points and multiple local minima. We discuss a small instance in Example 8.

First, let us define the problem of *structured low-rank approximation*. Here we are given a linear subspace $\mathcal{L} \subset \mathbb{R}^{m\times n}$, often with $U \in \mathcal{L}$, and we wish to solve the restricted problem:

$$\text{minimize} \; ||\, X - U \,||^2 = \sum_{i=1}^{m}\sum_{j=1}^{n} \lambda_{ij}(x_{ij}-u_{ij})^2 \;\; \text{subject to} \;\; X \in \mathcal{L} \text{ and } \operatorname{rank}(X) \leq r. \tag{10}$$

A best-case scenario for $\Lambda = \mathbf{1}$ would be this: if U lies in $\mathcal{L}$ then so does the SVD solution U^* in (9). This happens for some subspaces $\mathcal{L}$, including symmetric and circulant matrices. However, most subspaces $\mathcal{L}$ do not enjoy this property, and finding the optimal solution of (10) is difficult even for $\Lambda = \mathbf{1}$. The article [9] studies this optimization problem for both generic and special subspaces $\mathcal{L}$. It rests on [4] and uses tools from algebraic geometry.

As before, our primary task is to compute the number of complex critical points of (10). Thus, we seek to find the Euclidean distance degree (ED degree) of the determinantal variety

$$\mathcal{L}_{\leq r} \quad := \quad \big\{\, X \in \mathcal{L} \,:\, \operatorname{rank}(X) \leq r \,\big\}.$$

This variety is always regarded as a subvariety of the matrix space $\mathbb{R}^{m\times n}$, and we use the Λ-weighted Euclidean distance coming from $\mathbb{R}^{m\times n}$. We write $\operatorname{EDdegree}_\Lambda(\mathcal{L}_{\leq r})$ for the Λ-weighted Euclidean distance degree of the variety $\mathcal{L}_{\leq r}$. Thus $\operatorname{EDdegree}_\Lambda(\mathcal{L}_{\leq r})$ is the number of complex critical points of the problem (10) for generic data matrices U. The importance of keeping track of the weights Λ was highlighted in [4, Example 3.2], for the seemingly harmless situation when $\mathcal{L}$ is the subspace of all symmetric matrices in $\mathbb{R}^{n\times n}$.

Of special interest are the *unit ED degree*, when $\Lambda = \mathbf{1}$ is the all-one matrix, and the *generic ED degree*, denoted $\mathrm{EDdegree}_{\mathrm{gen}}(\mathcal{L}_{\leq r})$, when the weight matrix Λ is generic. The generic ED degree is given by a formula that rests on intersection theory. See [4, Theorem 7.7] and Theorem 9 below. Indeed, choosing the positive weights λ_{ij} to be generic ensures that the projective closure of $\mathcal{L}_{\leq r}$ has transversal intersection with the isotropic quadric

$$\{\, X \in \mathbb{P}^{mn-1} \ : \ \sum_{i=1}^{m}\sum_{j=1}^{n} \lambda_{ij} x_{ij}^2 \, = \, 0 \,\}.$$

We next present two examples that illustrate the concepts above. These can then also serve as examples for Theorem 9, as seen by the `Macaulay2` calculation in Example 15.

Example 7. Let $m = n = 3$ and $\mathcal{L} \subset \mathbb{R}^{3\times 3}$ the 5-dimensional space of Hankel matrices:

$$X \,=\, \begin{bmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \\ x_2 & x_3 & x_4 \end{bmatrix}, \quad U \,=\, \begin{bmatrix} u_0 & u_1 & u_2 \\ u_1 & u_2 & u_3 \\ u_2 & u_3 & u_4 \end{bmatrix} \quad \text{and} \quad \Lambda \,=\, \begin{bmatrix} \lambda_0 & \lambda_1 & \lambda_2 \\ \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_2 & \lambda_3 & \lambda_4 \end{bmatrix}.$$

Our goal in (10) is to solve the following constrained optimization problem for $r = 1, 2$:

$$\text{minimize } \lambda_0(x_0-u_0)^2 + 2\lambda_1(x_1-u_1)^2 + 3\lambda_2(x_2-u_2)^2 + 2\lambda_3(x_3-u_3)^2 + \lambda_4(x_4-u_4)^2$$
$$\text{subject to } \mathrm{rank}(X) \leq r.$$

This can be rephrased as an unconstrained optimization problem. For instance, for rank $r = 1$, we get a one-to-one parametrization of $\mathcal{L}_{\leq 1}$ by setting $x_i = st^i$, and our problem is to

$$\text{minimize } \lambda_0(t-u_0)^2 + 2\lambda_1(st-u_1)^2 + 3\lambda_2(st^2-u_2)^2 + 2\lambda_3(st^3-u_3)^2 + \lambda_4(st^4-u_4)^2.$$

The ED degree is the number of critical points with $t \neq 0$. We consider three weight matrices:

$$\mathbf{1} \,=\, \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \qquad \Omega \,=\, \begin{bmatrix} 1 & 1/2 & 1/3 \\ 1/2 & 1/3 & 1/2 \\ 1/3 & 1/2 & 1 \end{bmatrix}, \qquad \Theta \,=\, \begin{bmatrix} 1 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 1 \end{bmatrix}.$$

Here Ω gives the usual Euclidean metric when $\mathcal{L}$ is identified with $\mathbb{R}^5$, and Θ arises from identifying $\mathcal{L}$ with the space of symmetric $2\times 2\times 2\times 2$-tensors, as in Section 4. We compute

$$\begin{array}{lll} \mathrm{EDdegree}_{\mathbf{1}}(\mathcal{L}_{\leq 1}) = 6, & \mathrm{EDdegree}_{\Omega}(\mathcal{L}_{\leq 1}) = 10, & \mathrm{EDdegree}_{\Theta}(\mathcal{L}_{\leq 1}) = 4, \\ \mathrm{EDdegree}_{\mathbf{1}}(\mathcal{L}_{\leq 2}) = 9, & \mathrm{EDdegree}_{\Omega}(\mathcal{L}_{\leq 2}) = 13, & \mathrm{EDdegree}_{\Theta}(\mathcal{L}_{\leq 2}) = 7. \end{array}$$

In both cases, Ω exhibits the generic behavior, so $\mathrm{EDdegree}_{\mathrm{gen}}(\mathcal{L}_{\leq r}) = \mathrm{EDdegree}_{\Omega}(\mathcal{L}_{\leq r})$. We refer to [9, Sections 3 and 4] for larger Hankel matrices and formulas for their ED degrees. $\diamond$

Example 8. Let $m = n = 3, r = 1$ but now take $\mathcal{L} = \mathbb{R}^{3\times 3}$, so this is just the weighted rank-one approximation problem for 3×3-matrices. We know from [4, Example 7.10] that $\text{EDdegree}_{\text{gen}}(\mathcal{L}_{\leq 1}) = 39$. We take a circulant data matrix and a circulant weight matrix:

$$U = \begin{bmatrix} -59 & 11 & 59 \\ 11 & 59 & -59 \\ 59 & -59 & 11 \end{bmatrix} \quad \text{and} \quad \Lambda = \begin{bmatrix} 9 & 6 & 1 \\ 6 & 1 & 9 \\ 1 & 9 & 6 \end{bmatrix}.$$

This instance has 39 complex critical points. Of these, 19 are real, and 7 are local minima:

$$\begin{bmatrix} 0.0826 & 2.7921 & -1.5452 \\ 2.7921 & 94.3235 & -52.2007 \\ -1.5452 & -52.2007 & 28.8890 \end{bmatrix}, \begin{bmatrix} -52.2007 & 28.8890 & -1.5452 \\ 2.7921 & -1.5452 & 0.0826 \\ 94.3235 & -52.2007 & 2.7921 \end{bmatrix}, \begin{bmatrix} -52.2007 & 2.7921 & 94.3235 \\ 28.8890 & -1.5452 & -52.2007 \\ -1.5452 & 0.0826 & 2.7921 \end{bmatrix},$$

$$\begin{bmatrix} -29.8794 & 36.2165 & -27.2599 \\ -32.7508 & 39.6968 & -29.8794 \\ 39.6968 & -48.1160 & 36.2165 \end{bmatrix}, \begin{bmatrix} -48.1160 & 36.2165 & 39.6968 \\ 36.2165 & -27.2599 & -29.8794 \\ 39.6968 & -29.8794 & -32.7508 \end{bmatrix}, \begin{bmatrix} -29.8794 & -32.7508 & 39.6968 \\ 36.2165 & 39.6968 & -48.1160 \\ -27.2599 & -29.8794 & 36.2165 \end{bmatrix},$$

$$\begin{bmatrix} -25.375 & -25.375 & -25.375 \\ -25.375 & -25.375 & -25.375 \\ -25.375 & -25.375 & -25.375 \end{bmatrix}.$$

The first three are the global minima. The last matrix is the local minimum where the objective function has the largest value: note that each entry equals $-203/8$. The entries of the first six matrices are algebraic numbers of degree 10 over $\mathbb{Q}$. For instance, the two upper left entries 0.0826 and -48.1160 are among the four real roots of the irreducible polynomial

$$\begin{gathered} 164466028468224x^{10} + 27858648335954688x^9 + 1602205386689376672x^8 + 7285836260028875412x^7 \\ -219872893604668041427 2x^6 - 1485453269038009814315 2x^5 + 2688673091228371095762316x^4 \\ +446120944551158886226785 87x^3 - 4135008044571245731933710 6x^2 \\ +2703912949904311688967477 5x - 1977632463563766878765625. \end{gathered}$$

Thus, the critical ideal in $\mathbb{Q}[x_{11}, x_{12}, \ldots, x_{33}]$ is not prime. It is the intersection of six maximal ideals. Their degrees over $\mathbb{Q}$ are $1, 2, 6, 10, 10, 10$, for a total of $39 = \text{EDdegree}_{\text{gen}}(\mathcal{L}_{\leq 1})$. $\diamond$

Explicit formulas are derived in [9, Section 3] for $\text{EDdegree}_{\text{gen}}(\mathcal{L}_{\leq r})$ when $\mathcal{L}$ is a generic subspace of $\mathbb{R}^{m\times n}$. This covers the four cases that arise by pairing affine subspaces or linear subspaces with either unit weights or generic weights. One important feature of determinantal varieties is they are not complete intersections. This implies that their ED degrees are much smaller than suggested by the upper bound in Proposition 3. In order to deal with such situations, we need the following algebro-geometric formula for ED degrees.

Theorem 9. *If X meets both the hyperplane at infinity and the isotropic quadric transversally, then EDdegree(X) equals the sum of the polar degrees of the projective closure of X.*

We shall explain all the terms used in this theorem. First of all, the *projective closure* of our affine variety $X \subset \mathbb{C}^n$ is its Zariski closure in complex projective space $\mathbb{P}^n$, which we also denote by X. Algebraically, $\mathbb{P}^n$ is obtained from $\mathbb{C}^n$ by adding one homogenizing coordinate x_0. We identify the affine space $\mathbb{C}^n$ with the open subset $\{x \in \mathbb{P}^n : x_0 \neq 0\}$. Its set complement $\{x \in \mathbb{P}^n : x_0 = 0\} \simeq \mathbb{P}^{n-1}$ is the *hyperplane at infinity* inside $\mathbb{P}^n$. The hypersurface $\{x \in \mathbb{P}^{n-1} : \sum_{i=1}^n x_i^2 = 0\}$ is called the *isotropic quadric*. It lives in the hyperplane at infinity and it has no real points. The hypothesis in Theorem 9 means that the intersection of X with these two loci is reduced and has the expected dimension.

Theorem 9 appears in [4, Proposition 6.10]. The hypothesis is stated precisely in precise terms in [4, equation (6.4)]. It holds for all X after a general linear change of coordinates.

For the projective variety X, one considers the ED problem for its affine cone in $\mathbb{R}^{n+1}$. The data vector now equals $u = (u_0, u_1, \ldots, u_n)$, and the augmented Jacobian is redefined so as to respect the fact that all polynomials are homogeneous. The general formula for this matrix and the homogeneous critical ideal appears in [4, equation (2.7)].

For a curve $X \subset \mathbb{P}^2$ with defining polynomial $f(x_0, x_1, x_2)$, the augmented Jacobian is

$$\mathcal{AJ} \;=\; \begin{pmatrix} u_0 & u_1 & u_2 \\ x_0 & x_1 & x_2 \\ \partial f/\partial x_0 & \partial f/\partial x_1 & \partial f/\partial x_2 \end{pmatrix},$$

and the homogeneous critical ideal in $\mathbb{R}[x_0, x_1, x_2]$ is computed as follows:

$$\mathcal{C}_{X,u} \;=\; \big\langle\, f,\, \det(\mathcal{AJ})\,\big\rangle \,:\, \big(\langle \partial f/\partial x_0, \partial f/\partial x_1, \partial f/\partial x_2 \rangle \cdot (x_1^2 + x_2^2)\,\big)^{\infty}. \tag{11}$$

The critical points are given by the variety $\mathcal{V}(\mathcal{C}_{X,u})$ in $\mathbb{P}^2$, whose cardinality is $\mathrm{EDdegree}(X)$. The factor $(x_1^2 + x_2^2)$ in the saturation step (11) is the isotropic quadric. It is needed whenever the hypothesis of Theorem 9 is not satisfied. Namely, it removes any extraneous component that may arise from non-transversal intersection of the curve X with the isotropic quadric.

Example 10 (Cardioid)**.** We consider the homogeneous version of the cardioid in Example 1:

$$f \;=\; (x_1^2 + x_2^2 + x_0 x_2)^2 \,-\, x_0^2(x_1^2 + x_2^2). \tag{12}$$

The projective curve $X = \mathcal{V}(f)$ has three singular points, namely that at the origin $\mathcal{V}(x_1, x_2)$ in $\mathbb{C}^2 = \{x_0 \neq 0\}$ and the two points in the isotropic quadric $\mathcal{V}(x_1^2 + x_2^2)$ in $\mathbb{P}^1 = \{x_0 = 0\}$.

The homogenous critical ideal $\mathcal{C}_{X,u}$ is generated by three cubics, and it defines seven points in $\mathbb{P}^2$. Hence the projective cardioid X has $\mathrm{EDdegree}(X) = 7$. This is also the ED degree of the affine cardioid in (4) but only after a linear change of coordinates. That change can be fairly modest: if we replace x_1 by $2x_1$ in (4) then the ED degree jumps from 3 to 7. $\diamond$

We now explain what the polar degrees of a variety $X \subset \mathbb{P}^n$ are. Points h in the dual projective space $(\mathbb{P}^n)^\vee$ represent hyperplanes $\{x \in \mathbb{P}^n : h_0 x_0 + \cdots + h_n x_n = 0\}$. We are interested in all pairs (x, h) in $\mathbb{P}^n \times (\mathbb{P}^n)^\vee$ such that x is a nonsingular point of X and h is tangent to X at x. The Zariski closure of this set is the *conormal variety* $N_X \subset \mathbb{P}^n \times (\mathbb{P}^n)^\vee$.

It is known that N_X has dimension $n-1$, and if X is irreducible then so is N_X. The image of N_X under projection onto the second factor is the dual variety $X^\vee$. The role of $x \in \mathbb{P}^n$ and $h \in (\mathbb{P}^n)^\vee$ can be swapped. The following biduality relations [5, §I.1.3] hold:

$$N_X = N_{X^\vee} \quad \text{and} \quad (X^\vee)^\vee = X.$$

The conormal variety is an object of algebraic geometry that offers the theoretical foundations for various aspects of duality in optimization, including primal-dual algorithms.

Example 11. For a plane curve $X = \mathcal{V}(f)$ in $\mathbb{P}^2$, the conormal variety N_X is a curve in $\mathbb{P}^2 \times (\mathbb{P}^2)^\vee$. Its ideal is derived from the ideal that is generated by f and the 2×2 minors of

$$\begin{pmatrix} h_0 & h_1 & h_2 \\ \partial f/\partial x_0 & \partial f/\partial x_1 & \partial f/\partial x_2 \end{pmatrix}$$

By saturation, we remove singularities and points on the isotropic quadric, to arrive at $\mathcal{C}_{X,u}$.

For instance, if f is the homogeneous cardioid in (12) then $X^\vee$ is the cubic defined by

$$16h_0^3 - 27h_0h_1^2 - 24h_0^2h_2 - 15h_0h_2^2 - 2h_2^3.$$

The ideal of N_X has ten minimal generators. In addition to the above generators of bidegrees $(4,0)$ and $(0,3)$, we find the quadric $x_0h_0 + x_1h_1 + x_2h_2$ of bidegree $(1,1)$, three cubics of bidegree $(2,1)$ like $x_1^2h_1 - 3x_2^2h_1 - x_0x_1h_2 + 4x_1x_2h_2$, and four cubics of bidegree $(1,2)$. ⋄

We now finally come to the polar degrees. To this end, we consider the cohomology ring of the product of two projective spaces which serves as our primal-dual ambient space:

$$H^*(\mathbb{P}^n \times (\mathbb{P}^n)^\vee, \mathbb{Z}) \;=\; \mathbb{Z}[s,t]/\langle s^{n+1}, t^{n+1}\rangle.$$

The class of the conormal variety N_X in this cohomology ring is a binary form of degree $n+1 = \operatorname{codim}(N_X)$ whose coefficients are nonnegative integers:

$$[N_X] \;=\; \delta_1(X)s^nt \,+\, \delta_2(X)s^{n-1}t^2 \,+\, \delta_3(X)s^{n-2}t^3 \,+\, \cdots \,+\, \delta_n(X)st^n.$$

The coefficients $\delta_i(X)$ of this binary form are the *polar degrees* of X.

Remark 12. The polar degrees satisfy $\delta_i(X) = \#(N_X \cap (L \times L'))$, where $L \subset \mathbb{P}^n$ and $L' \subset (\mathbb{P}^n)^\vee$ are general linear subspaces of dimensions $n+1-i$ and i respectively. This geometric interpretation implies that $\delta_i(X) = 0$ for $i < \operatorname{codim}(X^\vee)$ and for $i > \dim(X)+1$. Moreover, the first and last polar degree are the classical degrees for the dual pair of varieties:

$$\delta_i(X) = \text{degree}(X) \text{ for } i = \dim(X)+1 \text{ and } \delta_i(X) = \text{degree}(X^\vee) \text{ for } i = \operatorname{codim}(X^\vee). \quad (13)$$

Example 13. Let $X \subset \mathbb{P}^2$ be the cardioid in (12). The curve $N_X \subset \mathbb{P}^2 \times (\mathbb{P}^2)^\vee$ has the class

$$[N_X] \;=\; \text{degree}(X^\vee)\cdot s^2t \;+\; \text{degree}(X)\cdot st^2 \;=\; 3\cdot s^2t \,+\, 4\cdot st^2.$$

Thus the polar degrees of the cardiod are 3 and 4. Their sum 7 is the ED degree. ⋄

Example 14. Let X be a general surface of degree d in $\mathbb{P}^3$. Its dual $X^\vee$ is a surface of degree $d(d-1)^2$ in $(\mathbb{P}^3)^\vee$. The conormal variety N_X is a surface in $\mathbb{P}^3 \times (\mathbb{P}^3)^\vee$, with class

$$[N_X] \;=\; d(d-1)^2\, s^3t \;+\; d(d-1)\, s^2t^2 \;+\; d\, st^3.$$

The sum of the three polar degrees equals $\text{EDdegree}(X) = d^3 - d^2 + d$; see Proposition 3. ⋄

Theorem 9 allows us to compute the ED degree for many interesting varieties, e.g. using Chern classes [4, Theorem 5.8]. This is relevant for applications in machine learning [6] which rest on low-rank approximation of matrices and tensors with special structure [9].

Example 15 (Determinantal varieties)**.** Let $X_r \subset \mathbb{P}^{m^2-1}$ be the variety of $m \times m$ matrices $x = (x_{ij})$ of rank $\leq r$. By [10], the conormal variety N_X is cut out by nice matrix equations:

$$N_X \;=\; \big\{(x,h) \in \mathbb{P}^{m^2-1} \times \mathbb{P}^{m^2-1} \,:\, \mathrm{rank}(x) \leq r,\; \mathrm{rank}(h) \leq m-r,\; x\cdot h = 0 \text{ and } h \cdot x = 0\big\}.$$

In particular, the duality relation $(X_r)^\vee = X_{m-r}$ holds among determinantal varieties. Typing the above formula into `Macaulay2`, we compute the polar degrees for $r=1$ and $m=3$:

```
QQ[x11,x12,x13,x21,x22,x23,x31,x32,x33,h11,h12,h13,h21,h22,h23,h31,h32,h33,
   Degrees=> {{1,0},{1,0},{1,0},{1,0},{1,0},{1,0},{1,0},{1,0},{1,0},
              {0,1},{0,1},{0,1},{0,1},{0,1},{0,1},{0,1},{0,1},{0,1}}];
x = matrix {{x11,x12,x13},{x21,x22,x23},{x31,x32,x33}};
h = matrix {{h11,h12,h13},{h21,h22,h23},{h31,h32,h33}};
I = minors(2,x) + minors(3,h) + minors(1,x*h) + minors(1,h*x);
isPrime(I), codim(I), degree I
multidegree(I)
```

The code starts with the bigraded coordinate ring of $\mathbb{P}^8 \times \mathbb{P}^8$. It verifies that N_X has codimension 9 and that `I` is its prime ideal. The last command computes the polar degrees:

$$[N_X] \;=\; 3s^8t \,+\, 6s^7t^2 \,+\, 12s^6t^3 \,+\, 12s^5t^4 + 6s^4t^5. \tag{14}$$

After verifying (13), one concludes that $\mathrm{EDdegree}(X_1) = 3+6+12+12+6 = 29$. Indeed, after changing coordinates, the EDdegree for 3×3-matrices of rank 1 equals 39. We saw this already in Example 8, where 39 critical points were found by a numerical computation.

The primal-dual set-up of the conormal varieties allows for a very elegant formulation of the critical equations. We now assume that X is an irreducible variety defined by homogeneous polynomials in n variables. Thus X is an affine cone in $\mathbb{C}^n$. Its dual $Y = X^\vee$ is the affine cone over the dual of the projective variety given by X. Thus Y is also an affine cone in $\mathbb{C}^n$. In this setting, the conormal variety N_X is viewed as an affine variety of dimension n in $\mathbb{C}^{2n}$. The homogeneous ideals of these cones are precisely those discussed above.

Theorem 16. *The ED problems for X and Y coincide, and we have* $\mathrm{EDdegree}(X) = \mathrm{EDdegree}(Y)$. *Given a general data point $u \in \mathbb{R}^n$, the critical equations for this problem are:*

$$(x,h) \,\in\, N_X \quad \text{and} \quad x+h = u. \tag{15}$$

Proof. See [4, Theorem 5.2]. □

It is instructive to verify Theorem 16 for Example 15. For any data matrix u of size $m\times m$, the sum in (15) is a special decomposition of u, namely x of rank r plus h of rank $m-r$. It arises from zeroing out complementary singular values σ_i in the Eckhart-Young Theorem.

In general, there is no free lunch. The difficulty lies in computing the ideal of the conormal variety N_X. However, this should be thought of as a preprocessing step, to be carried out only once per model X. If an efficient presentation of N_X is available, our task is is to solve the system $x+h=u$ of n linear equations in $2n$ coordinates for the n-dimensional space N_X.

The discussion so far was restricted to the Euclidean norm. But, we can measure distances in $\mathbb{R}^n$ with any other norm $||\cdot||$. Our optimization problem (1) extends naturally:

$$\text{minimize } ||x-u|| \text{ subject to } x \in X. \tag{16}$$

The unit ball $B = \{x \in \mathbb{R}^n : ||x|| \leq 1\}$ is a convex body. Conversely, every centrally symmetric convex body B defines a norm, and we can paraphrase (16) as follows:

$$\text{minimize } \lambda \text{ subject to } \lambda \geq 0 \text{ and } (u+\lambda B) \cap X \neq \emptyset. \tag{17}$$

If the boundary of B is smooth and algebraic then we express the critical equations as a polynomial system. This is derived as before, but we now replace the first row of the augmented Jacobian matrix $\mathcal{AJ}$ with the gradient of the map $\mathbb{R}^n \to \mathbb{R}$, $x \mapsto ||x-u||$.

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