Tropical Geometry Homework 3

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1

The paper arXiv:07083847 by M. Vigeland answers this question. In Theorem 7.1, the author exhibits a particular subdivision of the Newton polytope of such a cubic surface, such that any tropical surface with this subdivision contains at least 27 tropical lines, and a general one contains exactly 27 lines. The subdivision is depicted in Figure 18. By inspection of this figure, I computed that this polyhedral complex has 20 vertices, 64 edges, 72 2-cells, and 27 3-cells; so a corresponding tropical variety is a polyhedral complex with 27 0-cells, 72 1-cells, and 64 2-cells.

2

I used gfan to calculate the tropicalization of $G_{2,6}$. The defining polynomials of this variety give a homogeneous prime ideal, and so we can compute its tropical variety as

```
./gfan_tropicalstartingcone | ./gfan_tropicaltraverse
Q[x12, x13, x14, x15, x16, x23, x24, x25, x26, x34, x35, x36,
x45, x46, x56]
{x12 x34 - x13 x24 + x14 x23,
x12 x35 - x13 x25 + x15 x23,
x12 x36 - x13 x26 + x16 x23,
x12 x45 - x14 x25 + x15 x24,
x12 x46 - x14 x26 + x16 x24,
x12 x56 - x15 x26 + x16 x25,
x13 x45 - x14 x35 + x15 x34,
```

```
x13 x46 - x14 x36 + x16 x34,
x13 x56 - x15 x36 + x16 x35,
x14 x56 - x15 x46 + x16 x45,
x23 x45 - x24 x35 + x25 x34,
x23 x46 - x24 x36 + x26 x34,
x23 x56 - x25 x36 + x26 x35,
x24 x56 - x25 x46 + x26 x45,
x34 x56 - x35 x46 + x36 x45
}
```

It is a 9-dimensional polyhedral complex with lineality space of dimension 6; modding out by this lineality space produces a polyhedral fan with 1 0-dimensional face, 25 rays, 105 2-dimensional cones, and 105 3-dimensional cones. The full output is included at the end of this document. This example is explained in detail in Anders Jensen's Users Manual for gfan.

Next, we construct a lift of the weight vector $w = e_{12} + e_{34} + e_{56}$ in the classical Grassmannian $G_{2,6}$; then by the Fundamental Theorem of Tropical Geometry, we can conclude that w lies in the tropical Grassmannian. Recall that $G_{2,6}$ is the variety of relations among 2×2 subdeterminants of a 2×6 matrix; thus it suffices to construct a 2×6 matrix with entries in $\mathbb{Q}(t)$ whose 2×2 minors have the appropriate valuations. Some fiddling produces, for example,

$$\begin{pmatrix} t & 2t & 1 & 1 & 1+t & 1+2t \\ 1 & 1 & t & 2t & 1 & 1 \end{pmatrix}$$

and the corresponding point

$$(x_{12}, x_{13}, x_{14}, x_{15}, x_{16}, x_{23}, x_{24}, x_{25}, x_{26}, x_{34}, x_{35}, x_{36}, x_{45}, x_{46}, x_{56})$$

in the classical projective variety $G_{2,6}$ is

$$(-t, t^2-1, 2t^2-1, -1, -1-t, 2t^2-1, 4t^2-1, -1+t, -1, t,$$
$$1-t-t^2, 1-t-2t^2, 1-2t-2t^2, 1-2t-4t^2, -t).$$

3

The surface in question is the secant variety of a Veronese surface, or, alternatively, the space of symmetric $2 \times 2 \times 2 \times 2$ tensors of tensor rank at most 2. To compute the implicit equation of the given hypersurface, we compute the intersection

$$(u^4+x^4-a, u^3v+x^3y-b, u^2v^2+x^2y^2-c, uv^3+xy^3-d, v^4+y^4-e) \cap k[a,b,c,d,e].$$

This is easily done in Macaulay2, producing the ideal $(c^3 - 2bcd + ad^2 + b^2e - ace)$. This is the ideal given by the 3×3 minors of the 4×4 flattenings of a symmetric $2 \times 2 \times 2 \times 2$ tensor.

```
Macaulay 2, version 1.0
with packages: Classic, Core, Elimination, IntegralClosure,
LLLBases, Parsing, PrimaryDecomposition, SchurRings,
TangentCone

i1 : R = QQ[a,b,c,d,e,u,v,x,y];

i2 : I = ideal(u^4 + x^4 - a, u^3*v + x^3*y - b, u^2*v^2
+x^2*y^2-c, u*v^3+x*y^3-d, v^4+y^4 - e);

o2 : Ideal of R

i3 : J = eliminate(I,{u,v,x,y})

            3                2    2
o3 = ideal(c  - 2b*c*d + a*d  + b e - a*c*e)

o3 : Ideal of R
```

This ideal is homogeneous and prime; so we feed it to gfan as follows to compute its tropical variety.

```
./gfan_tropicalstartingcone | ./gfan_tropicaltraverse
Q[a,b,c,d,e]
{c^3 - 2bcd + ad^2 + b^2e - ace}

_application PolyhedralFan
_version 2.2
_type PolyhedralFan

AMBIENT_DIM
5

DIM
4

LINEALITY_DIM
```

```
2

RAYS
-1 0 0 0 0      # 0
2 1 0 0 0       # 1
0 -1 0 0 0      # 2
-3 -2 -1 0 0    # 3
-2 -1 -1 0 0    # 4
4 3 2 0 0       # 5

N_RAYS
6

LINEALITY_SPACE
0 1 2 3 4
1 0 -1 -2 -3

ORTH_LINEALITY_SPACE
0 0 1 -2 1
0 1 0 -3 2
1 0 0 -4 3

F_VECTOR
1 6 9

CONES
{}      # Dimension 2
{0}     # Dimension 3
{1}
{2}
{3}
{4}
{5}
{1 2}   # Dimension 4
{0 3}
{0 4}
{0 5}
{1 4}
{1 5}
{3 4}
```

```
{2 3}
{2 5}

MAXIMAL_CONES
{1 2}	# Dimension 4
{0 3}
{0 4}
{0 5}
{1 4}
{1 5}
{3 4}
{2 3}
{2 5}

PURE
1

MULTIPLICITIES
1	# Dimension 4
1
1
1
1
1
1
1
1
```

Let L be the linear subspace of $\mathbb{R}^5$ spanned by the rows of the matrix

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 \end{pmatrix},$$

so that L is precisely the lineality space of the above tropical variety. Modding out by L, taking negatives (to account for `gfan` using the max-plus instead of min-plus) and intersecting with a sphere about the origin, we obtain a graph on 6 vertices with 9 edges which is isomorphic to the prism

$K_3 \times K_2$, with vertices

$$\begin{aligned} v_0 &= (1,0,0,0,0) \\ v_1 &= (-2,-1,0,0,0) \\ v_2 &= (0,1,0,0,0) \\ v_3 &= (3,2,1,0,0) \\ v_4 &= (2,1,1,0,0) \\ v_5 &= (-4,-3,-2,0,0). \end{aligned}$$

See the diagram below.

Next, note that the tropical parametrization

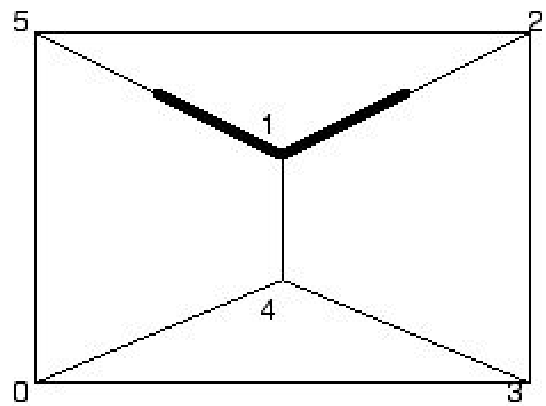
$$(U^4 \oplus X^4, U^3V \oplus X^3Y, U^2V^2 \oplus X^2Y^2, UV^3 \oplus XY^3, V^4 \oplus Y^4)$$

is precisely the tropical secant variety T of the linear (in the usual sense) subspace L, and M. Develin's paper arXiv:0405115 outlines how to compute these. First observe that given $x \in \mathbb{R}^5$, we have $x \in T$ iff $x + L \subseteq T$. So we may mod out by L, and assume (as in the `gfan` output above) that the last two coordinates of x are 0. Now, following Develin, the points $x = (x_0, x_1, x_2, 0, 0) \in T$ are precisely those such that the upper envelope of the polytope in $\mathbb{R}^2$ constructed as the convex hull of the 5 points $(0, x_0), (1, x_1), (2, x_2), (3, 0), (4, 0)$ contains two facets meeting all these points. Then it is straightforward case-checking to show that the points in $T \cap \mathbb{R}^3$ are given by

$$\{(x_0,0,0,0,0) : x_0 \leq 0\} \cup \{(x_0,x_1,0,0,0) : x_0, x_1 < 0, x_1 \geq x_0/2\}$$

$$\cup \{(x_0,x_1,x_2,0,0) : x_0,x_1,x_2 < 0, x_1 = (x_0+x_2)/2, x_2 \geq x_1/2\}.$$

This is a subset of the graph described above; it occupies the vertex v_1 and the two "half-edges" from v_1 and v_2 and from v_1 to v_5. Thus it is a subset of the graph, and even has a natural complex structure with 3 0-cells and 2 bounded 1-cells; but it is not a subcomplex. We do note, however, that the (tropical) secant variety of the tropicalization is contained in the tropicalization of the (classical) secant variety.



4 Appendix: `gfan` output for $G_{2,6}$

```
_application PolyhedralFan
_version 2.2
_type PolyhedralFan

AMBIENT_DIM
15

DIM
9

LINEALITY_DIM
6

RAYS
-1 0 0 0 0 0 0 0 0 0 0 0 0 0 0  # 0
0 -1 0 0 0 0 0 0 0 0 0 0 0 0 0  # 1
0 0 -1 0 0 0 0 0 0 0 0 0 0 0 0  # 2
1 1 1 1 0 0 0 0 0 0 0 0 0 0 0   # 3
0 0 0 -1 0 0 0 0 0 0 0 0 0 0 0  # 4
-1 -1 0 0 0 -1 0 0 0 0 0 0 0 0 0       # 5
0 0 0 0 0 -1 0 0 0 0 0 0 0 0 0  # 6
-1 0 -1 0 0 0 -1 0 0 0 0 0 0 0 0       # 7
0 0 0 0 0 0 -1 0 0 0 0 0 0 0 0  # 8
1 1 1 1 0 1 1 1 0 0 0 0 0 0 0   # 9
1 0 0 0 0 1 1 1 0 0 0 0 0 0 0   # 10
```

```
-1 0 0 -1 0 0 0 -1 0 0 0 0 0 0 0        # 11
0 0 0 0 0 0 0 -1 0 0 0 0 0 0 0  # 12
-1 -1 -1 0 0 -1 -1 0 0 -1 0 0 0 0 0     # 13
0 0 0 0 0 -1 -1 0 0 -1 0 0 0 0 0        # 14
0 -1 -1 0 0 0 0 0 0 -1 0 0 0 0 0        # 15
0 0 0 0 0 0 0 0 0 -1 0 0 0 0 0  # 16
1 1 1 1 0 1 1 1 0 1 1 0 0 0 0   # 17
1 1 0 0 0 1 1 1 0 1 1 0 0 0 0   # 18
1 1 1 1 0 1 0 0 0 1 1 0 0 0 0   # 19
0 1 0 0 0 1 0 0 0 1 1 0 0 0 0   # 20
-1 -1 0 -1 0 -1 0 -1 0 0 -1 0 0 0 0     # 21
0 0 0 0 0 -1 0 -1 0 0 -1 0 0 0 0        # 22
0 -1 0 -1 0 0 0 0 0 0 -1 0 0 0 0        # 23
0 0 0 0 0 0 0 0 0 0 -1 0 0 0 0  # 24

N_RAYS
25

LINEALITY_SPACE
0 0 0 0 0 1 1 1 1 1 1 1 1 1 1
0 0 0 0 1 0 0 0 1 0 0 1 0 1 1
0 0 0 1 0 0 0 1 0 0 1 0 1 0 1
0 0 1 0 0 0 1 0 0 1 0 0 1 1 0
0 1 0 0 0 0 -1 -1 -1 0 0 0 -1 -1 -1
1 0 0 0 0 0 0 0 0 -1 -1 -1 -1 -1 -1

ORTH_LINEALITY_SPACE
0 0 0 0 0 0 0 0 0 0 1 -1 -1 1 0
0 0 0 0 0 0 0 0 0 1 0 -1 -1 0 1
0 0 0 0 0 0 0 1 -1 0 0 0 -1 1 0
0 0 0 0 0 0 1 0 -1 0 0 0 -1 0 1
0 0 0 0 0 1 0 0 -1 0 0 -1 -1 1 1
0 0 0 1 -1 0 0 0 0 0 0 0 -1 1 0
0 0 1 0 -1 0 0 0 0 0 0 0 -1 0 1
0 1 0 0 -1 0 0 0 0 0 0 -1 -1 1 1
1 0 0 0 -1 0 0 0 -1 0 0 0 -1 1 1

F_VECTOR
1 25 105 105
```

```
CONES
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{0}     # Dimension 7
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{4}
{5}
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{7}
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{22}
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{24}
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{7 8}
{2 7}
{3 8}
{4 8}
{0 9}
```

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MAXIMAL_CONES
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PURE
1

MULTIPLICITIES
1       # Dimension 9
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```
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