Math 274: Tropical Geometry, Homework 5

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(with solutions also by Claudiu Raicu and Dustin Cartwright)

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1. Show that every balanced one-dimensional fan in $\mathbb{R}^3$ is the tropicalization of a space curve in $\mathbb{C}^3$. More precisely, let $v_1, v_2, \ldots, v_n$ be integer vectors in $\mathbb{Z}^3$ such that $v_1 + v_2 + \ldots + v_n = 0$, and let Σ be the union of the n rays $\mathbb{R}_{\geq 0} v_i$. Construct a prime ideal $I \subset \mathbb{C}[x, y, z]$ with $T(I) = \Sigma$.

Proof. [**Solution due to Claudiu Raicu**] Consider two polynomials $f, g \in \mathbb{C}[X^{\pm}]$ such that the spherical graphs they determine intersect transversely at the points where the vectors v_i intersect the sphere. This implies that $\mathcal{T}(f)$ and $\mathcal{T}(g)$ intersect transversely ([1]) at each point and $\mathcal{T}(f) \cap \mathcal{T}(g) = \Sigma$. Using the Transverse Intersection Lemma (lemma 3.2, corollary 3.3 of [1]) we get that $\Sigma = \mathcal{T}(f, g)$, so it suffices to prove that we can choose f, g so that (f, g) is prime. We can then take $I = (f, g) \cap \mathbb{C}[X]$.

We now show how to choose f, g: clearly $\mathcal{T}(f), \mathcal{T}(g)$ depend only on the monomials in f, g, so we can fix these sets of monomials (and denote them A, B). Abusing notation we write $f = \sum_{\alpha \in A} a_\alpha X^\alpha$, $g = \sum_{\beta \in B} b_\beta X^\beta$ where a_α, b_β are indeterminates. Since the generic fiber of the natural map f : Spec $\mathbb{C}[X^{\pm}, a_\alpha, b_\beta] \to$ Spec $\mathbb{C}[a_\alpha, b_\beta]$ is integral and we are working over an algebraically closed field it follows that there is an open set where the fiber of f is integral, and since closed points are dense in schemes of finite type over fields we can find a closed point in that open set. This will give the desired f, g. $\square$

2. Determine the maximal Barvinok rank of any 5x5-matrix whose entries are 0 or 1. Do you have a conjecture for $n \times n$-matrices over $\{0, 1\}$?

The maximal Barvinok rank of any 5x5 matrix whose entries are 0 or 1 is 5. First we know that for any $n \times n$ matrix, $A = [a_{i,j}]$, over $\{0, 1\}$ the Barvinok rank is less than or equal to n because,

$$[a_{i,j}] = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \odot \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} 1 \\ \vdots \\ 1 \\ 0 \end{pmatrix} \odot \begin{pmatrix} a_{n,1} & a_{n,2} & \cdots & a_{n,n} \end{pmatrix}.$$

Next there exists a 5x5 matrix over $\{0, 1\}$ that has Barvinok rank of 5; consider the following matrix,

$$\begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \odot \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} \odot \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \end{pmatrix}.$$

Because of the placement of the zeros in this example it is necessary to use these general types of rank 1 matrices to get this matrix–note none of the rank 1 matrices can have a zero appearing off the diagonal. In fact for any $n \times n$ matrix the maximal Barvinok rank is n by our first observation and the fact that any $n \times n$ matrix with zeros down the diagonal and ones everywhere else has Barvinok rank of n.

3. A *polytrope* is a subset P of tropical projective space $\mathbb{TP}^d$ such that P is both tropically convex and classically convex. What is the maximal number of classical vertices of 4-dimensional polytrope ($d = 4$)?

According to Joswig and Kulas [4], who took this from Develin and Strumfels [2], each $d-$polytope has at most $\binom{2d}{d}$ classical vertices, and this bound is sharp (Proposition 16). Joswig and Kulas even go on to find such a polytope that would almost surely attain the maximal number of vertices in section 4.3 of their paper, [4]. Thus the maximal number of classical vertices of a 4-dimensional polytrope is 70.

4. Consider a general arrangement of n tropical hyperplanes in $\mathbb{TP}^d$. How many components are there in the complement of this arrangement?

[Solution due to Dustin Cartwright]

The complement of a general arrangement of n tropical hyperplanes in $\mathbb{TP}^d$ has $\binom{n+d}{d}$ components. Each hyperplane is defined by a homogeneous linear from in $d + 1$ variables. The union of the hyperplanes is defined by the product of these polynomials, which is a homogenous polynomial of degree n. The tropical hypersurface is dual to a polyhedral subdivision of the Newton polytope of the product of the polynomials, which is a d-dimensional simplex scaled by n. Because the arrangement is generic, each lattice point of the Newton polytope is a vertex of the subdivision, which therefore has cardinality $\binom{n+d}{d}$. The components of the complement are dual to the vertices of the subdivision, and thus there $\binom{n+d}{d}$ components.

5. Consider the family of plane cubic curves of the special form

$$c_1x^2y + c_2xy^2 + c_3x^2 + c_4xy + c_5y^2 + c_6x + c_7y = 0.$$

(a) Compute the Newton polytope of the discriminant of this family.

(b) How many rational curves of the above special form pass through five general points in the complex projective plane?

(c) Draw a picture that demonstrates the tropical solution to this curve counting problem, i.e., pick ve general points in the plane $\mathbb{TP}^2$ and determine all tropical curves of genus zero passing through your points.

[Solution due to Dustin Cartwright]

(a) I used the following Singular code to compute the discriminant:

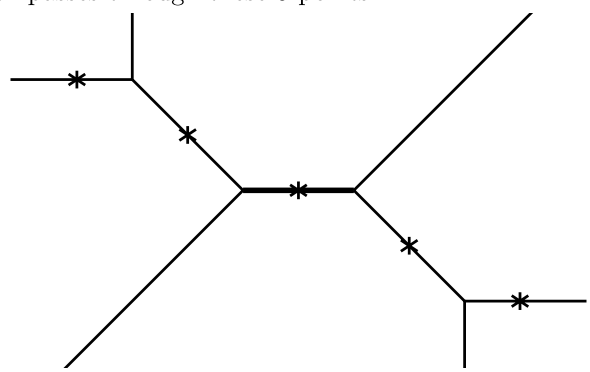
```
ring r = 0, (x, y, a, b, c, d, e, f, g), dp;
poly F = ax2y+bxy2+cx2+dxy+ey2+fx+gy;
ideal I = F, diff(F,x), diff(F,y);
ideal J = std(I);
ideal K = eliminate(J, x*y);
poly D = K[1];
LIB "poly.lib";
```

```
newtonDiag(D);
exit;
```

The discriminant's degree is 12. By a computation in Polymake, the f-vector of the Newton polytope is $(127, 65, 45, 12)$.

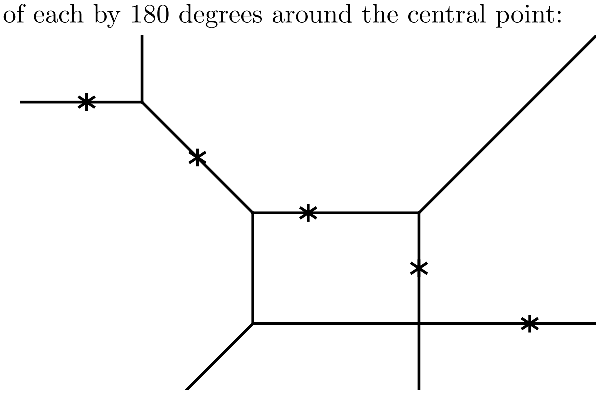
(b) A generic cubic polynomial has genus 1. It is rational if and only if it has a singularity. Passing through 5 general points imposes 5 independent linear conditions on the coefficients of the polynomial. Thus the coefficients of rational curves passing through 5 general points are the intersection of the discriminant with 5 hyperplanes. Since the discriminant has degree 12, this intersection is 12 points in $\mathbb{P}^6$. In other words, there are 12 rational curves of the specified form passing through 12 general points.

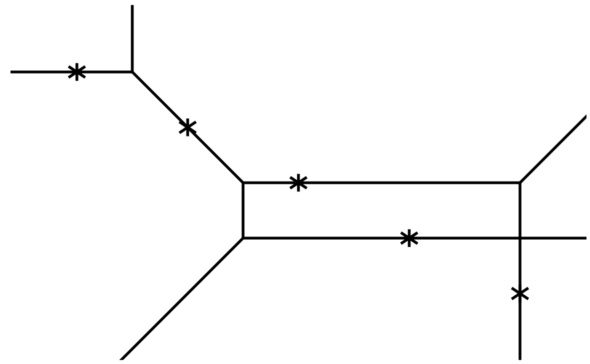
(c) I chose 5 points which each differ from the previous one by the previous one by the vector $(2, -1)$. Since none of my curves have edges parallel to this vector, these points are generic. There is one centrally symmetric curve which passes through these 5 points:

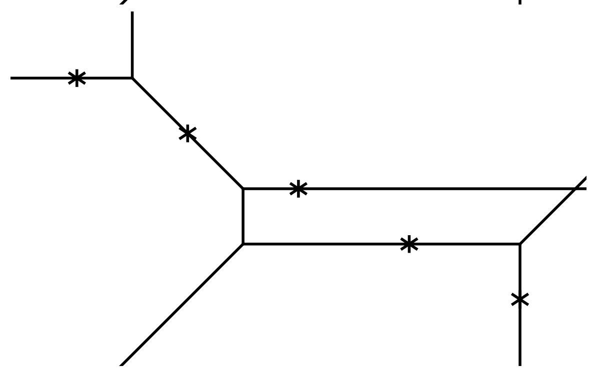


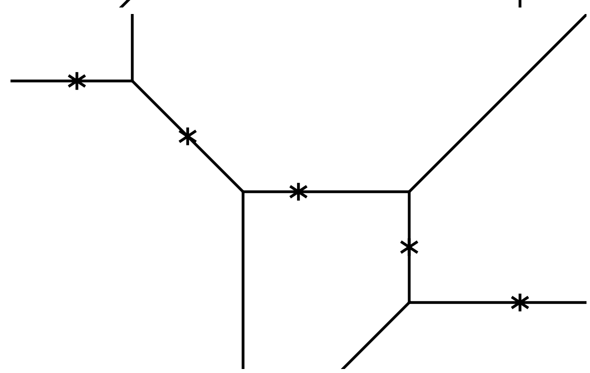
In this graph, there are two vertices with multiplicity 2, on either end of the double line, so this tropical curve counts for 4 classical solutions.

In addition, the following 4 curves pass through the 5 points, as do the rotations of each by 180 degrees around the central point:









These curves, together with their rotations, total 8 classical solutions.

6. Write a short essay ($\leq$ 2 pages): *How are cluster algebras related to tropical geometry?*

In [6], Speyer and Williams describe the tropicalization of the totally positive part of the Grassmannian $Gr_{k,n}$ and how this is related to cluster algebras. The relationship isn't surprising because the properties of cluster algebras make them relevant to studying total positivity and homogenous spaces such as Grassmannians. In fact, Scott shows in [5] that the coordinate rings of Grassmannians have natural cluster algebra structures.

According to [3], for any positive integer n, a cluster algebra, A, of rank n is a commutative ring with unit and no zero divisors, equipped with a distinguished family of generators called cluster variables. These variables have an exchange property and it is interesting to note that any cluster variable can be represented as a subtraction-free rational expression in the variables of any given cluster because of this exchange relation. The main example of a cluster algebra of an arbitrary rank n, provided by [3], is the homogenous coordinate ring $\mathbb{C}[Gr_{2,n+3}]$ of the Grassmannian of 2-dimensional subspaces in $\mathbb{C}^{n+3}$. This ring, $\mathbb{C}[Gr_{2,n+3}]$, is generated by the Plucker coordinates $[ij]$ for $1 \leq i < j \leq n+3$, subject to the relations $[ik][jl] = [ij][kl] + [il][jk]$,

$(i < j < k < l)$. Next, Fomin and Zelevinksky describe how we can consider each $[ij]$ to be the sides and diagonals of the convex $(n+3)$-gon: the sides are scalars and the diagonals are the cluster variables. Then the clusters are the maximal families of pairwise non-crossing diagonals, so they are in a natural bijection with the triangulation of this polygon. Also the monomials in the variables of any given cluster form a linear basis in $\mathbb{C}[Gr_{2,n+3}]$.

Speyer and Williams introduce the totally positive part of the tropicalization of an arbitrary affine variety, which is an object that has the structure of a polyhedral fan, [6]. They pay particular attention to the Grassmannian, denoting the resulting fan $\text{Trop}^+ Gr_{k,n}$. For instance, the fan of $\text{Trop}^+ Gr_{2,n}$ is combinatorially equivalent to the normal fan of the associahedron and since $S(A)$–an abstract simplicial complex–is isomorphic to the dual of the associahedron, A_{n-3}, it follows that $\text{Trop}^+ Gr_{2,n}$ is combinatorially the cone on $S(A)$. They also consider $\text{Trop}^+ Gr_{3,6}$, $\text{Trop}^+ Gr_{3,7}$ which relate to D_4 and E_6 respectively. This is interesting because it is similar to the results of Scott (and Fomin and Zelevinsky) who showed that the Grassmannian has a natural cluster algebra structure which is of the types A_{n-3}, D_4, and E_6 for $Gr_{2,n}, Gr_{3,6}$, and $Gr_{3,7}$, [5] and [3].

Using the information above and other relevant material, Speyer and Williams develop the following observation: *When A is the coordinate ring of a Grassmannian, embed $SpecA$ in affine space by the variables $X \sqcup C$. Then $Trop^+ SpecA$ is a fan with lineality space of dimension $|C|$. After taking the quotient by this lineality space, we get a simplical fan abstractly isomorphic to the cone over $S(A)$. [6]* Speyer and Williams make a special note that this does not quite hold for an arbitrary cluster algebra of finite type, their conjecture however is that: *Let A be a cluster algebra of nite type over $\mathbb{R}$ and $S(A)$ its associated cluster complex. If the lineality space of $Trop^+ SpecA$ has dimension $|C|$ then $Trop^+ SpecA$ modulo its lineality space is a simplicial fan abstractly isomorphic to the cone over $S(A)$. If the condition on the lineality space does not hold, the resulting fan is a coarsening of the cone over $S(A)$. [6]* Thus we see that cluster algebras are nicely related to tropical geometry through the Grassmannian.

References

[1] T. Bogart, A. Jensen, D. Speyer, B. Sturmfels and R. Thomas, Computing tropical varieties, Journal of Symbolic Computation 42 (2007) 5473.

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[5] J. Scott: Grassmannians and cluster algebras. (arXiv: 0311148).

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