

Math 55: Discrete Mathematics

UC Berkeley, Fall 2011
Homework # 5, due Wednesday, February 22

5.1.4 Let $P(n)$ be the statement that $1^3 + 2^3 + \cdots + n^3 = (n(n+1)/2)^2$ for the positive integer n .

- a) What is the statement $P(1)$?
 - b) Show that $P(1)$ is true.
 - c) What is the induction hypothesis?
 - d) What do you need to prove in the inductive step?
 - e) Complete the inductive step.
 - f) Explain why these steps show that this formula is true for all positive integers n .
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- a) $P(1)$ is the statement $1^3 = ((1(1+1)/2)^2$.
 - b) This is true because both sides of the equation evaluate to 1.
 - c) The induction hypothesis is the statement $P(k)$ for some positive integer k , that is, the statement $1^3 + 2^3 + \cdots + k^3 = (k(k+1)/2)^2$.
 - d) Assuming that $P(k)$ holds, we need to show that $P(k+1)$ holds, that is, we need to derive the equation $1^3 + 2^3 + \cdots + k^3 + (k+1)^3 = ((k+1)(k+2)/2)^2$ from the equation in (c).
 - e) We add $(k+1)^3$ to both the left hand side and the right hand side of the equation in (c). This shows that the left hand side in (d) is equal to $(k(k+1)/2)^2 + (k+1)^3$. By expanding and factoring, we find that this expression equals $((k+1)(k+2)/2)^2$. Hence we have shown that the left hand side of the equation in (d) equals the right hand side of the equation in (d).
 - f) We have carried out both the basis step and the inductive step. The principle of mathematical induction now ensures that $P(n)$ is true for all positive integers n .

5.1.6 *Prove that $1 \cdot 1! + 2 \cdot 2! + \cdots + n \cdot n! = (n + 1)! - 1$ whenever n is a positive integer.*

We use mathematical induction. In the basis step, for $n = 1$, the equation states that $1 \cdot 1! = (1 + 1)! - 1$, and this is true because both sides of the equation evaluate to 1. For the inductive step, we assume that $1 \cdot 1! + 2 \cdot 2! + \cdots + k \cdot k! = (k + 1)! - 1$ for some positive integer k . We add $(k + 1)(k + 1)!$ to the left hand side to find that

$$1 \cdot 1! + 2 \cdot 2! + \cdots + (k + 1) \cdot (k + 1)! = (k + 1)! - 1 + (k + 1)(k + 1)!$$

The right hand side equals $(k + 1)!(k + 2) - 1 = (k + 2)! - 1$. This establishes the desired equation also for $k + 1$, and we are done by the principle of mathematical induction.

5.1.10 a) *Find a formula for*

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{n(n + 1)}$$

by examining the values of this expression for small values of n .

b) *Prove the formula you conjectured in part (a).*

(a) By evaluating the sum for $n = 1, 2, 3, 4, 5, \dots$, we are led to conjecture that the following equation holds for all positive integers n :

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{n(n + 1)} = \frac{n}{n + 1}. \quad (1)$$

(b) We use mathematical induction. The basis step is $n = 1$. Here both sides of the equation are equal to $1/2$, so the claim holds. For the inductive step, we assume that (1) is true for $n = k$. We add $\frac{1}{(k+1)(k+2)}$ to both sides of this equation. Then the right hand side becomes

$$\frac{k}{k + 1} + \frac{1}{(k + 1)(k + 2)} = \frac{k(k + 2) + 1}{(k + 1)(k + 2)} = \frac{k + 1}{k + 2}.$$

Hence the left hand side of (1) for $n = k + 1$ equals the right hand side of (1) for $n = k + 1$. This completes the proof by induction.

5.1.18 *Prove that $n! < n^n$ for all integers $n \geq 2$, using the six suggested steps.*

Let $P(n)$ be the propositional function $n! < n^n$.

- a) The statement $P(2)$ says that $2! = 2$ is less than $2^2 = 4$.
- b) This statement is true because 4 is larger than 2.
- c) The inductive hypothesis states that $P(k)$ holds for some integer $k \geq 2$.
- d) We need to prove that $k! < k^k$ implies $(k+1)! < (k+1)^{k+1}$.
- e) Given that $k! < k^k$ holds, easily seen inequalities imply

$$(k+1)! = k! \cdot (k+1) < k^k(k+1) < (k+1)^k \cdot (k+1) = (k+1)^{k+1}.$$

- f) We have carried out both the basis step and the inductive step. The principle of mathematical induction now ensures that $P(n)$ is true for all integers $n \geq 2$.

5.1.32 *Prove that 3 divides $n^3 + 2n$ whenever n is a positive integer.*

We use mathematical induction. For $n = 1$, the assertion says that 3 divides $1^3 + 2 \cdot 1$, which is indeed the case, so the basis step is fine. For the inductive step, we assume that 3 divides $k^3 + 2k$ for some positive integer k . Hence there exists an integer l such that $3l = k^3 + 2k$. A computation shows

$$(k+1)^3 + 2(k+1) = (k^3 + 2k) + 3(k^2 + k + 1).$$

The right hand is divisible by 3. This is evident for the second summand, and it is the induction hypothesis for the first summand. Hence we have proved that 3 divides $(k+1)^3 + 2(k+1)$. This complete the inductive step, and hence the assertion follows.

5.1.54 *Use mathematical induction to show that given a set of $n+1$ positive integers, none exceeding $2n$, there is at least one integer in this set that divides another integer in the set.*

Let $P(n)$ be the following propositional function: given a set of $n+1$ positive integers, none exceeding $2n$, there is at least one integer in this set that divides another integer in the set. The proposition $P(1)$ is true because there is only one set of $1+1$ positive integers none exceeding $2 \cdot 1$. This set is $\{1, 2\}$ and it contains an integer, namely 1, that divides the other integer, namely 2. This verifies the basis step in our proof by mathematical induction.

For the inductive step we assume that $P(k)$ is true for some positive integer k . To prove $P(k+1)$, we consider a set S of $k+2$ positive

integers none exceeding $2k + 2$. If $S \cap \{2k + 1, 2k + 2\}$ has cardinality 0 or 1 then we apply the induction hypothesis to $S \setminus \{2k + 1, 2k + 2\}$ to conclude that this set contains a dividing pair of integers.

Hence we are left with the case that $2k + 1$ and $2k + 2$ are both in S and $S \setminus \{2k + 1, 2k + 2\}$ consists of k positive integers of size at most $2k$ that pairwise don't divide each other. If $k + 1$ is in S then we are done because $k + 1$ divides $2k + 2$. Suppose therefore that $k + 1 \notin S$. Then we replace S by the set $S' = (S \setminus \{2k + 2\}) \cup \{k + 1\}$. The new set S' is covered by the previous case, so it contains a divisible pair. If that pair does not involve $k + 1$ then it is also in S . If it involves $k + 1$ then this means that some $l \in S \setminus \{k + 1\}$ divides l . That l must also divide $2k + 2$ and hence S contains a divisible pair. This completes the inductive step and hence the proof.

5.2.4 *Let $P(n)$ be the statement that a postage of n cents can be formed using just 4-cent stamps and 7-cent stamps. Prove that $P(n)$ is true for $n \geq 18$, using the six suggested steps.*

We prove this using strong induction. The basis step is to check that $P(18)$, $P(19)$, $P(20)$ and $P(21)$ hold. This seen from the identities

$$18 = 4 + 7 + 7, \quad 19 = 4 + 4 + 4 + 7, \quad 20 = 4 + 4 + 4 + 4 + 4, \quad 21 = 7 + 7 + 7.$$

For the inductive step, we assume that $P(j)$ holds for all integers j with $18 \leq j \leq k$ where $k \geq 21$. To realize $k + 1$ cents, we first realize $k - 3$ cents using 4-cent stamps and 7-cent stamps. This is possible by the inductive hypothesis, since $k - 3 \geq 18$. Now add one more 4-cent stamp to realize $k + 1$ cents. This completes the induction step and it hence proves the assertion.

5.2.10 *Assume that a chocolate bar consists of n squares arranged in a rectangular pattern. The entire bar, or a smaller rectangular piece of the bar, can be broken along a vertical or a horizontal line separating the squares. Assuming that only one piece can be broken at a time, determine how many breaks you must successively make to break the bar into n separate squares use strong induction to prove your answer.*

We claim that the number of needed breaks is $n - 1$. We shall prove this for all positive integers n using strong induction. The basis step $n = 1$ is clear. In that case we don't need to break the chocolate at all, we can just eat it. Suppose now that $n \geq 2$ and assume the assertion is true for all rectangular chocolate bars with fewer than n

squares. Then we break the chocolate into two pieces of size m and $n - m$ where $1 \leq m < n$. By the induction hypotheses, the bar with m pieces requires $m - 1$ breaks and the bar with $n - m$ squares requires $n - m - 1$ breaks. Thus the original chocolate bar requires $1 + (m - 1) + (n - m - 1)$ breaks. This number equals $n - 1$, as required.

5.2.26 Suppose that $P(n)$ is a propositional function. Determine for which nonnegative integers n the statement $P(n)$ must be true if

- a) $P(0)$ is true; for all nonnegative integers n , if $P(n)$ is true then $P(n + 2)$ is true.
 - b) $P(0)$ is true; for all nonnegative integers n , if $P(n)$ is true then $P(n + 3)$ is true.
 - c) $P(0)$ and $P(1)$ are true; for all non-negative integers n , if $P(n)$ and $P(n + 1)$ are true then $P(n + 2)$ is true.
 - d) $P(0)$ is true; for all non-negative integers n , if $P(n)$ is true then $P(n + 2)$ and $P(n + 3)$ are true.
- a) The statement $P(n)$ is true for all nonnegative integers n that are even.
 - b) The statement $P(n)$ is true for all nonnegative integers n that are divisible by 3.
 - c) The statement $P(n)$ is true for all nonnegative integers n .
 - d) The statement $P(n)$ is true for all nonnegative integers n with $n \neq 1$, since every such n is expressible as a sum of 2's and 3's.

5.3.4 Find $f(2)$, $f(3)$, $f(4)$, and $f(5)$ if f is defined recursively by $f(0) = f(1) = 1$ and for $n = 1, 2, \dots$

- a) $f(n + 1) = f(n) - f(n - 1)$,
 - b) $f(n + 1) = f(n)f(n - 1)$,
 - c) $f(n + 1) = f(n)^2 + f(n - 1)^2$,
 - d) $f(n + 1) = f(n)/f(n - 1)$.
- a) 0, -1, -2, -3
 - b) 1, 1, 1, 1
 - c) 2, 5, 29, 866
 - d) 1, 1, 1, 1

5.3.6 Determine whether each of these proposed definitions is a valid recursive definition of a function f from the set of all nonnegative integers to the set of integers. If f is well defined, find a formula for $f(n)$ when n is a nonnegative integer and prove that your formula is valid.

- a) $f(0) = 1, f(n) = -f(n - 1)$ for $n \geq 1$,
- b) $f(0) = 1, f(1) = 0, f(2) = 2, f(n) = 2f(n - 3)$ for $n \geq 3$,
- c) $f(0) = 0, f(1) = 1, f(n) = 2f(n + 1)$ for $n \geq 2$,
- d) $f(0) = 0, f(1) = 1, f(n) = 2f(n - 1)$ for $n \geq 1$,
- e) $f(0) = 2, f(n) = f(n - 1)$ if n is odd and $n \geq 1$ and $f(n) = 2f(n - 2)$ if $n \geq 2$.

- a) The function is well-defined and given by $f(n) = (-1)^n$.
- b) The function is well-defined and given by $f(n) = 0$ if $n \equiv 1 \pmod{3}$ and $f(n) = 2^{\lfloor (n-1)/2 \rfloor}$ otherwise.
- c) The function is not well-defined because the definition of $f(n)$ involves the value at $n + 1$.
- d) The function is well-defined and given by $f(n) = 2^{\lfloor (n+3)/2 \rfloor}$.

5.3.8 Give a recursive definition of the sequence $[a_n]$, $n = 1, 2, 3, \dots$ if

- a) $a_n = 6n$,
- b) $a_n = 2n + 1$,
- c) $a_n = 10^n$,
- d) $a_n = 5$.

- a) $a_1 = 6$ and $a_n = a_{n-1} + 6$ for $n \geq 2$.
- b) $a_1 = 3$ and $a_n = a_{n-1} + 2$ for $n \geq 2$.
- c) $a_1 = 1$ and $a_n = 10a_{n-1}$ for $n \geq 2$.
- d) $a_1 = 5$ and $a_n = a_{n-1}$ for $n \geq 2$.

5.3.12 Let f_n denote the n th Fibonacci number. Prove that $f_1^2 + f_2^2 + \dots + f_n^2 = f_n f_{n+1}$ when n is a positive integer.

We prove this by induction on n . The statement is true for $n = 1$ because $1^2 = 1 \cdot 1$, it is true for $n = 2$ because $1^2 + 1^2 = 1 \cdot 2$, and it is true for $n = 3$ because $1^2 + 1^2 + 2^2 = 2 \cdot 3$. This takes care of the

basis step. For the inductive step, suppose it is true for $n = k$, and consider the left hand for $n = k + 1$. We find

$$f_1^2 + f_2^2 + \cdots + f_k^2 + f_{k+1}^2 = f_k f_{k+1} + f_{k+1}^2 = f_{k+1}(f_k + f_{k+1}).$$

The last expression equals $f_{k+1} f_{k+2}$ by the recursive definition of the Fibonacci sequences. This verifies the equation for $n = k + 1$. We have thus completed both the basis step and the inductive step, and hence the claim follows by the principle of mathematical induction.

5.3.20** Give a recursive definition of the functions \max and \min , so that that $\max(a_1, a_2, \dots, a_n)$ and $\min(a_1, a_2, \dots, a_n)$ are the maximum and the minimum of the n numbers a_1, a_2, \dots, a_n respectively.

$$\max(a_1, \dots, a_n) = \begin{cases} a_1 & \text{if } n = 1 \text{ (base case)} \\ a_n & \text{if } n > 1 \text{ and} \\ & a_n > \max(a_1, \dots, a_{n-1}) \\ \max(a_1, \dots, a_{n-1}) & \text{otherwise} \end{cases}$$

$$\min(a_1, \dots, a_n) = \begin{cases} a_1 & \text{if } n = 1 \text{ (base case)} \\ a_n & \text{if } n > 1 \text{ and} \\ & a_n < \min(a_1, \dots, a_{n-1}) \\ \min(a_1, \dots, a_{n-1}) & \text{otherwise} \end{cases}$$

5.3.51 Find these values of Ackermann's function

a) $A(2, 3)$

b) $A(3, 3)$

The Ackermann function is defined recursively by

$$A(m, n) = 2n \text{ if } m = 0;$$

$$A(m, n) = 0 \text{ if } m \geq 1 \text{ and } n = 0;$$

$$A(m, n) = 2 \text{ if } m \geq 1 \text{ and } n = 1;$$

$$A(m, n) = A(m-1, A(m, n-1)) \text{ if } m \geq 1 \text{ and } n \geq 2.$$

We have $A(2, 2) = A(1, A(2, 1)) = A(1, 2) = A(0, A(1, 1)) = A(0, 2) = 4$. Also, we have $A(1, 1) = 2$, $A(1, 2) = A(0, A(1, 1)) = A(0, 2) = 4$, $A(1, 3) = A(0, A(1, 2)) = A(0, 4) = 8$, and $A(1, 4) = A(0, A(1, 3)) = A(0, 8) = 16$. More generally, $A(1, n) = 2^n$. This is Exercise 50.

a) $A(2, 3) = A(1, A(2, 2)) = A(1, 4) = 16.$

b) $A(3, 3) = A(2, A(3, 2)) = A(2, A(1, A(3, 1))) = A(2, A(1, 2)) = A(2, 4) = A(1, A(2, 3)) = A(1, 16) = 2^{16} = 65536.$