Tropical geometry homework 1

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Everything here is done in the min convention.

1 Given a tropical polynomial $f(x) = \bigoplus_{i=0}^{d} a_i x^{\odot d-i}$, $a_0 = 0$, we wish to show that it can be uniquely written in the form $\bigodot_{i=1}^{d} x \oplus c_i$ for some multiset of constants c_i (uniqueness up to order of the factors).

Consider this function $g(x) := \bigodot_{i=1}^{d} x \oplus c_i$, which is a classical sum of the classical functions $\max\{x, c_i\}$. This factor $\max\{x, c_i\}$ has slope 1 for $x < c_i$ and slope 0 for $x > c_i$. Since slope is additive, the slope of g at some x not equal to any c_i is equal to the number of i such that $c_i > x$. Thus the graph of g determines the multiset of c_i uniquely: they can be read off as the discontinuities of the piecewise linear function, each discontinuity c occurring with multiplicity $\lim_{x\to c-} g'(x) - \lim_{x\to c+} g'(x)$, the difference of slopes on either side. Furthermore, we can write in the form of g any piecewise linear function whose slope is nonincreasing as x increases, taking values d at $x \ll 0$ to 0 at $x \gg 0$ and integral values everywhere.

So it remains to show that our tropical polynomial $f(x)$ satisfies these conditions. Classically $f(x)$ is a minimum of the classical linear functions $a_i + (d-i)x$, the ith of which is a line of slope i. The minimum of all these functions is piecewise linear and agrees with one of them, i.e. has integer slope, on every interval. The slope is nonincreasing, since any line part of which forms part of the graph of f must otherwise lie completely above it. Finally the slopes for $x \ll 0$ and $x \gg 0$ are respectively d and 0, since the linear functions dx resp. a_d dominate in those cases.

2 We derive this formula as a special case of a general procedure for extracting roots of tropical polynomials $f(x)$, namely, traversing the graph of $f(x)$ and noting roots at the discontinuities, multiplicity equal to the difference in slopes.

Let $f(x) = \bigoplus_{i=0}^{d} a_i x^{\odot d-i}$ be the tropical polynomial in question, $a_0 = -\infty$. Let l_i be the line which is the tropical graph of the ith term, $a_i x^{\odot d-i}$. We start traversing at x sufficiently small, so that $f(x) = x^{\odot d}$, i.e. on l_0. Suppose at some point during the traversal that we are on l_i. The line forming the next segment of the graph of f will be that line l_j with $j > i$ whose intersection has least x-coordinate, and at the intersection of these two lines we'll get a root $x = (a_j - a_i)/(j - i)$ of multiplicity $j - i$. (The way tie-breaking is performed is of no consequence to this.)

Carrying the procedure out in this case, our traversal starts on l_0. We then move to l_1 or l_2 or l_3; in the former case we might move to l_2 or l_3 next; in the second case we move to l_3; and once we're on l_3 we're done. This translates to

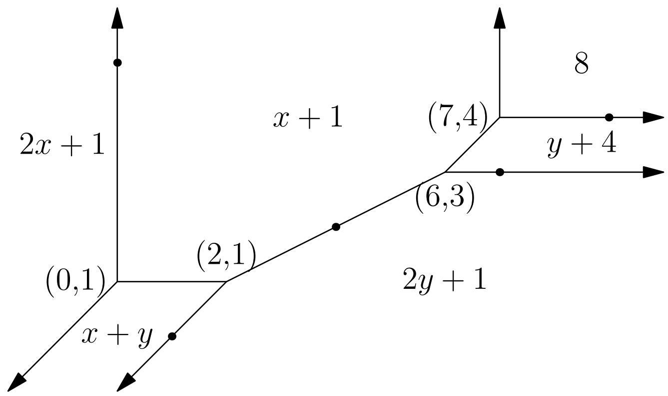

Figure 1: The unique quadratic curve

the following cubic formula.

$$\mathrm{roots}(f) = \begin{cases} c/3, c/3, c/3 & c/3 \le b/2, a \\ b/2, b/2, c-b & b/2 \le c/3, a \\ a, (c-a)/2, (c-a)/2 & a \le b/2, c/3; (c-a)/2 \le b-a \\ a, b-a, c-b & a \le b/2, c/3; b-a \le (c-a)/2 \end{cases}.$$

3 If eyeballing the problem and just coming up the requisite curve was good enough for Diane's minicourse, it should be good enough for here...

Of the four generic shapes of tropical quadratic curve in the plane. the one which looks most promising here is the one with a segment with direction (2,1), since several of the points look roughly lined up in that direction. Then we hit upon the curve of Figure 1.

To work back to the quadratic polynomial giving rise to this curve, we label the regions of the fan with the linear forms that are to be minimal there, starting by writing a constant (which we've chosen to make everything come out nonnegative) in the upper-rightmost region and working across edges from there. Thus we get

$$8 \oplus 4 \odot y \oplus 1 \odot y^{\odot 2} \oplus 1 \odot x \oplus 0 \odot x \odot y \oplus 1 \odot x^{\odot 2}$$

as the tropical quadratic cutting out this curve.

(Another way to get at this — Dan and Tony's method from Diane's course? — would have been to solve a classical lifting of this problem, in other words to find a linear relation among the vectors $(1, x, y, x^2, xy, y^2)$ where now x, y are classical variables for each of these five points, which can be quickly done with a format 6×6 determinant.)

4 Smooth cubic curves are dual to regular subdivisions of $3\Delta_2 = \mathrm{convex}\{(0,0),(0,3),(3,0)\}$ with precisely nine regions. Every region has volume at least $\frac{1}{2}$, since it contains

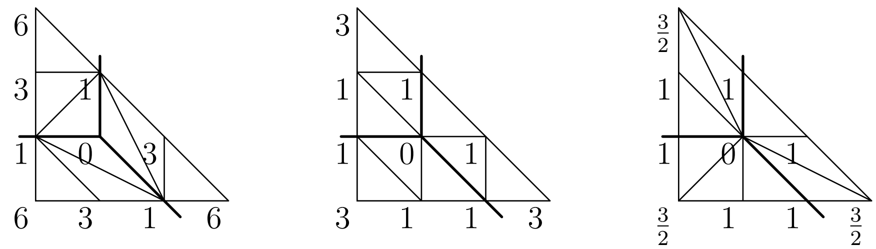

Figure 2: Regular triangulations with 3, 6, and 9 edges incident to the interior vertex.

a nondegenerate lattice triangle whose volume is half an integer 2×2 determinant and thus is at least $\frac{1}{2}$. But $\mathrm{vol}(3\Delta_2) = 9/2$, so if there are to be nine regions each must have area $\frac{1}{2}$. Thus each region must be a triangle, otherwise it'd have a triangulation with multiple triangles, and have larger area; also these triangles must have no lattice points in their interior or on their boundary, by Pick's theorem.

A bounded region of a tropical curve comes from an interior vertex in its dual subdivision. Since no lattice points of $3\Delta_2$ may be interior points or boundary points of any region, every interior lattice point is an interior vertex. Hence there's exactly one such, at $(1, 1)$.

The number of sides of the bounded region is the number of edges incident to this internal vertex in the dual. This number can be at most nine, one for each triangle, and must be at least three, since each triangle spans an angle strictly less than π at this internal vertex, so there must be at least three triangles to cover an angle of 2π. To exhibit that each number of edges 3 through 9 is possible, the three regular triangulations of Figure 2 have, from left to right, $3 \cdot 1$, $3 \cdot 2$, and $3 \cdot 3$ edges incident to their interior vertex, and by cutting into pieces along the thicker lines and reassembling the pieces we can get any admissible number of edges.

We draw the tropical curves dual to these three triangulations in Figure 3. Again, the remaining numbers of sides of the bounded region can be gotten by cutting the edges corresponding to the thick lines (i.e. those overlying the edges of the bounded region in the left diagram) and reassembling.

5 We use the Cunningham-Greene machinery for this.

The eigenvalue of A is then the least normalised weight of any cycle. Examining each of the $0!\binom{3}{1} + 1!\binom{3}{2} + 2!\binom{3}{3} = 8$ cycles, we find that the minimum normalised weight is $7/3 = \lambda(A)$ and is achieved by (1 2 3).

To find the eigenvector, we first normalise A to

$$B = -\lambda(A) \odot A = \begin{bmatrix} 5/3 & 5/3 & 8/3 \\ -4/3 & 2/3 & -1/3 \\ -4/3 & 2/3 & 5/3 \end{bmatrix}$$

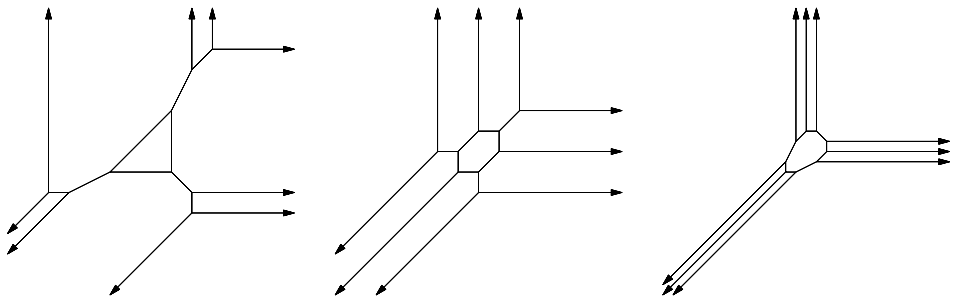

Figure 3: Tropical cubics with 3, 6, and 9 edges to the bounded region.

and then get the eigenspace as the tropical image of

$$B \oplus B^{\odot 2} \oplus B^{\odot 3} = \begin{bmatrix} 5/3 & 5/3 & 8/3 \\ -4/3 & 2/3 & -1/3 \\ -4/3 & 2/3 & 5/3 \end{bmatrix} + \begin{bmatrix} 1/3 & 7/3 & 4/3 \\ -5/3 & 1/3 & 1/3 \\ -2/3 & 1/3 & 1/3 \end{bmatrix} + \begin{bmatrix} 0 & 2 & 2 \\ -1 & 0 & 0 \\ -1 & 1 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 5/3 & 4/3 \\ -5/3 & 0 & -1/3 \\ -4/3 & 1/3 & 0 \end{bmatrix}$$

But in fact this matrix has rank 1: all the columns are tropical scalar multiples of one another. Thus, within the tropical projective plane $\mathbb{R}^3/(1,1,1)$ which we identify with $\mathbb{R}^2$, there is just one point in the eigenspace, namely $(4/3, -1/3)$; and the general tropical eigenvector is $(t + 4/3, t - 1/3, t)$.

Considering the (tropical) determinant as a polynomial in the entries, each of its monomials is the sum of the entries of several disjoint cycles. Since the cycle of minimum weight includes every element, the term it contributes to the determinant must be the minimum; that is, $\det A = 7$ tropically.

6 We'll determine these images elementarily.

The image of a general point $x = (x_1, x_2, x_3, x_4)$ under the matrix at left has ith coordinate $y_i = \min_j x_j + \delta_{ij}$, to use classical notation. Suppose x_j is a minimal component of x. Then $y_i = x_j$ unless perhaps $i = j$, for which $y_j = \min\{x_j + 1, \text{numbers} \geq x_j\}$. Thus $x_j \leq y_j \leq x_j + 1$, and every such value is achievable by appropriate choices of the other x_i. The image of the matrix thus consists of the union of all coordinate permutations of $\{(x, x, x, x') : x' \in [x, x+1]\}$. Projecting to $\mathbb{TP}^3$, we have the left of Figure 4, where the endpoints are $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$, $(-1, -1, -1)$.

The image of x under the matrix at right has ith coordinate $y_i = \min_j x_j + 1 - \delta_{ij}$. Suppose again x_j is a minimal component of x. Then $y_j = x_j$, and for other i, $y_i = \min\{x_i, x_j + 1\}$, since $x_j + 1$ is the minimum of the various $x_k + 1$. This can take any value y_i in the range $[x_j, x_j + 1]$, with a suitable choice of x_i. Note that $y_j = x_j$ remains the minimum component of y. Therefore the image of the matrix consists of all points y all of whose components are contained

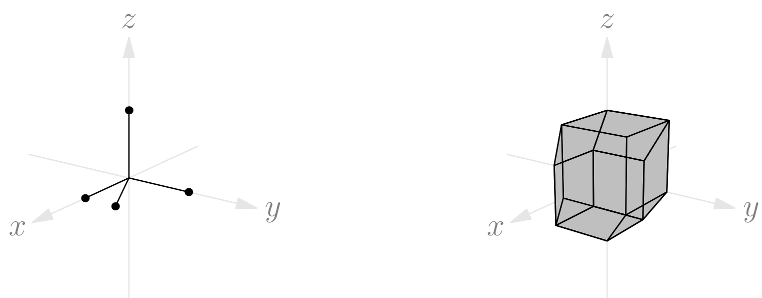

Figure 4: Left and right: the images of the respective matrices.

in $[y_j, y_j+1]$, where j is their least component. The projection to $\mathbb{TP}^3$ is drawn in the right of Figure 4: this is the Minkowski sum of the four line segments incident on the origin in the drawing at left.

7 I've dug up an example I had lying around from earlier, which is a computation of the tropical linear space associated to the matrix

$$\begin{bmatrix} 0 & 2 & 4 & 6 & 8 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

This can be computed as the tropical variety of the Plücker ideal of a lift of this matrix. The following input file represents this, taking the trivial lift t^i of the tropical entry i.

```
Q[t,x1,x2,x3,x4,x5]
{
t^11*x2-t^11*x3-2*t^10*x2+t^10*x3+t^10*x4+t^9*x3-t^9*x4+2*t^8*x2
-t^8*x4-t^8*x5-t^7*x2-t^7*x3+2*t^7*x5-t^6*x3+t^6*x4+t^5*x3+t^5*x4
-2*t^5*x5-t^4*x4+t^4*x5,
t^11*x1-t^11*x3-2*t^10*x1+t^10*x3+t^10*x4+2*t^8*x1+t^8*x3-2*t^8*x4
-t^8*x5-t^7*x1+t^7*x5-t^6*x3+t^6*x5-t^4*x3+2*t^4*x4-t^4*x5+t^3*x3
-t^3*x5-t^2*x4+t^2*x5,
-t^11*x1+t^11*x2+t^10*x1-t^10*x2+t^9*x1-t^9*x4-t^8*x2+t^8*x4-t^7*x1
+t^7*x5-t^6*x1+t^6*x2+t^6*x4-t^6*x5+t^5*x1-t^5*x5+t^4*x2-t^4*x4
-t^3*x2+t^3*x5-t^2*x4+t^2*x5+t*x4-t*x5,
t^10*x1-t^10*x2-t^9*x1+t^9*x3-t^8*x1+2*t^8*x2-t^8*x3+t^6*x1-t^6*x3
+t^5*x1-t^5*x5-t^4*x1-2*t^4*x2+t^4*x3+2*t^4*x5+t^2*x2+t^2*x3
-2*t^2*x5-t*x3+t*x5,
-t^8*x1+t^8*x2+2*t^7*x1-t^7*x2-t^7*x3-t^6*x2+t^6*x3-2*t^5*x1+t^5*x3
+t^5*x4+t^4*x1+t^4*x2-2*t^4*x4+t^3*x2-t^3*x3-t^2*x2-t^2*x3+2*t^2*x4
+t*x3-t*x4
}
```

Running this through `gfan_tropicalintersection --tplane` yields the following output.

```
_application PolyhedralFan
_version 2.2
_type PolyhedralFan

AMBIENT_DIM
6

DIM
5

LINEALITY_DIM
1

RAYS
0 -1 0 0 0 0 # 0
0 1 0 0 0 0 # 1
-1 3 1 0 0 0 # 2
0 0 -1 0 0 0 # 3
0 0 1 0 0 0 # 4
-1 5 3 1 0 0 # 5
-1 4 2 1 0 0 # 6
0 0 0 -1 0 0 # 7
0 0 0 1 0 0 # 8
0 -1 -1 -1 -1 0 # 9
0 1 1 1 1 0 # 10
0 0 0 0 -1 0 # 11
0 0 0 0 1 0 # 12

N_RAYS
13

LINEALITY_SPACE
0 1 1 1 1 1

ORTH_LINEALITY_SPACE
0 0 0 0 1 -1
0 0 0 1 0 -1
0 0 1 0 0 -1
0 1 0 0 0 -1
1 0 0 0 0 0

F_VECTOR
1 13 43 56 20
```

```
CONES
{} # Dimension 1
{0} # Dimension 2
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{7}
{10}
{11}
{1}
{2}
{4}
{5}
{6}
{8}
{9}
{12}
{0 3} # Dimension 3
{0 7}
{3 7}
{0 10}
{3 10}
{0 11}
{3 11}
{10 11}
{7 10}
{7 11}
{0 2}
{2 3}
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{6 7}
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{4 11}
{5 11}
{5 10}
{6 11}
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{7 9}
```

```
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{1 3 10}
{0 3 11}
```

```
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{0 7 10}
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{1 7 10}
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{4 7 10}
{3 7 11}
{3 7 12}
{3 8 11}
{4 7 11}
{0 3 7 12} # Dimension 5
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{3 7 9 11}
{0 3 7 9}
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{0 4 7 10}
{3 7 10 12}
{0 3 9 11}
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{0 4 10 11}
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{0 7 10 12}
{0 8 10 11}
{4 7 10 11}
{1 3 7 11}
{1 3 7 10}
{1 3 10 11}
{1 7 10 11}

MAXIMAL_CONES
{0 2 3} # Dimension 4
```

```
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{0 2 7}
{5 6 7}
{2 6 7}
{2 3 6}
{2 3 7}
{0 6 11}
{0 6 10}
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{5 6 11}
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{3 6 11}
{3 6 10}
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{5 7 10}
{0 3 7 12} # Dimension 5
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{0 7 10 12}
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{1 3 7 11}
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{1 3 10 11}
{1 7 10 11}

PURE
0
```

Per the manual, this does not remove the cones lying in the plane $t = 0$, and these we have to remove by hand. We find that these are all the cones involving the rays 1 4 8 9 12. In particular, none of the putative maximal cones of dimension 5 are contained in our tropical variety, and it is in fact pure; the true f-vector with these bad rays removed is 1 8 23 16.