This exam is closed book. Do not use any books, notes or
electronic devices. Please write in a blue note book, with your
name, the name of your GSI and your section time on the front.
Each of the 10 problems is worth 10 points, for a total of 100
points. Answers without justification will receive no credit.

(1) Suppose \( n \geq 1 \) is an integer.
   (a) How many functions are there from the set \( \{1, \ldots, n\} \)
to the set \( \{1, 2, 3\} \)?
   (b) How many of the functions in part (a) are injective?
   (c) How many of the functions in part (a) are surjective?

(2) Let \( G \) be a simple graph with at least two vertices. Show
that if \( G \) is not connected, then the complementary graph
\( \overline{G} \) is connected.

(3) Suppose each of the three vertices of a triangle is colored
at random either black or white, with equal probability for
each color. What is the expected number of edges of the
triangle that have both endpoints of the same color?

(4) Give an example of a relation \( R \) such that its transitive
closure \( R^* \) satisfies \( R^* = R \cup R^2 \cup R^3 \), but \( R^* \neq R \cup R^2 \).

(5) Recall that the Fibonacci numbers are defined by \( f_0 = 0 \),
\( f_1 = 1 \), and \( f_n = f_{n-2} + f_{n-1} \) for \( n \geq 2 \). Prove that

\[
f_0f_1 + f_1f_2 + \cdots + f_{2n-1}f_{2n} = f_{2n}^2
\]

for any positive integer \( n \).
(6) Find the set of all solutions \( x \) to the system of congruences

\[
x \equiv 2 \pmod{6} \quad \text{and} \quad x \equiv 3 \pmod{9}.
\]

(7) Suppose we randomly choose nonnegative integers \( x_1, x_2, x_3, \) and \( x_4 \) that solve the equation \( x_1 + x_2 + x_3 + x_4 = 8 \). Here we assume that each solution has an equal probability of being chosen. Given that at least one of \( x_1 \) and \( x_2 \) is equal to 1, what is the probability that \( x_1 = 1 \).

(8) Solve the recurrence relation

\[
a_n = 2a_{n-1} + a_{n-2} - 2a_{n-3}
\]

with initial conditions \( a_0 = 1, a_1 = 0, \) and \( a_2 = 7 \).

(9) Give a recursive definition of the set of bit strings that have the same number of zeros and ones.

(10) Use generating functions to determine the number of different ways 10 identical balloons can be given to four children if each child receives at least two balloons.

(Note: The children are distinguishable)