CONVEX ALGEBRAIC GEOMETRY

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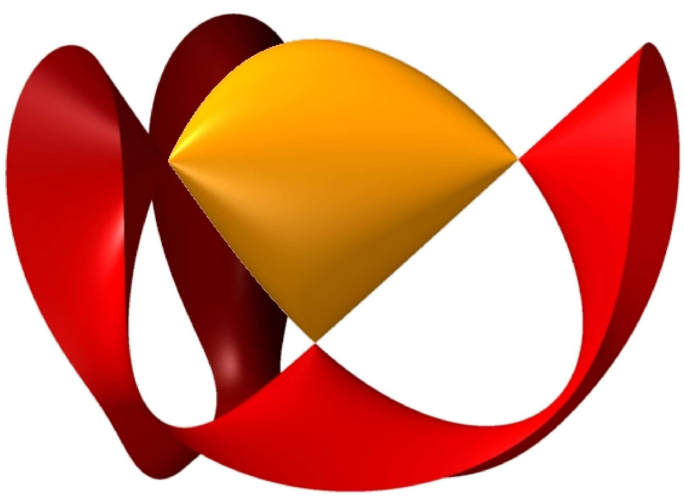

Multifocal Ellipses

Given m points $(u_1, v_1), \ldots, (u_m, v_m)$ in the plane $\mathbb{R}^2$, and a radius $d > 0$, their *m-ellipse* is the convex algebraic curve

$$\left\{(x,y) \in \mathbb{R}^2 \,:\, \sum_{k=1}^{m} \sqrt{(x-u_k)^2 + (y-v_k)^2} \;=\; d \right\}.$$

The 1-ellipse and the 2-ellipse are algebraic curves of degree 2.

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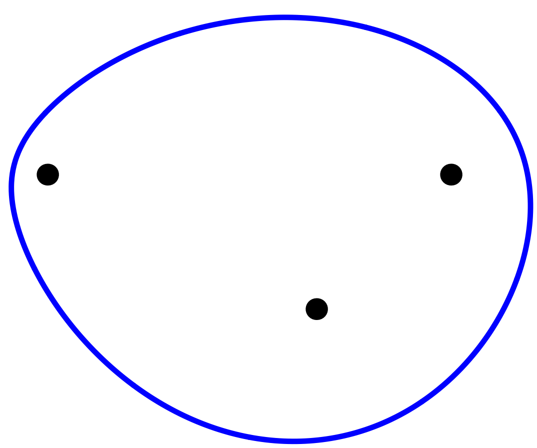
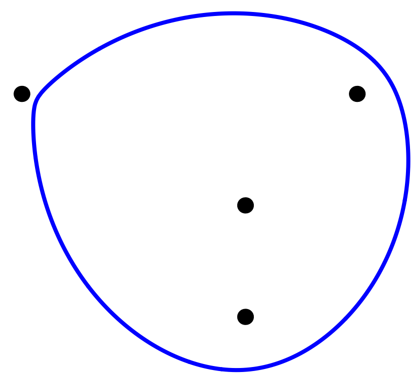
The 3-ellipse is an algebraic curve of degree 8:

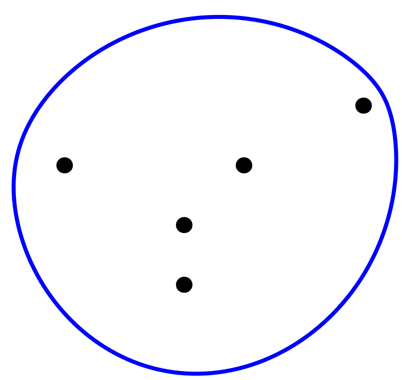


2, 2, 8, 10, 32, ...

The 4-ellipse is an algebraic curve of degree 10:

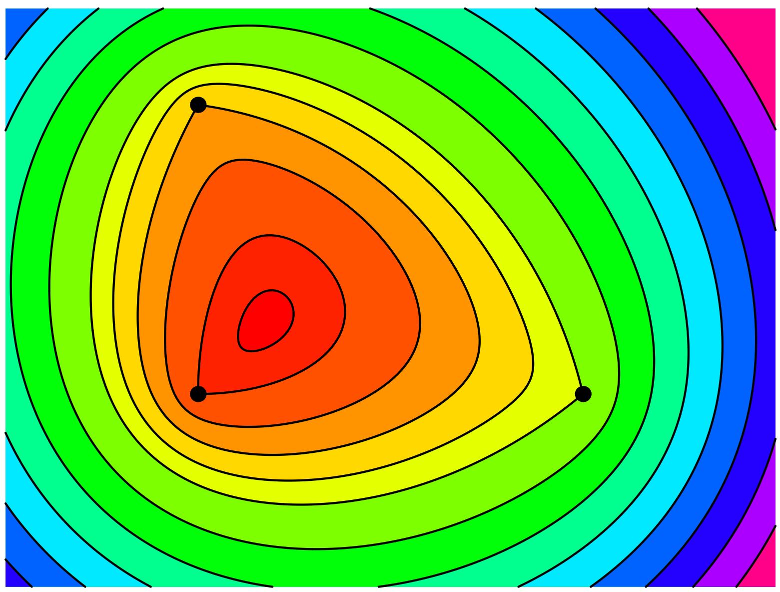


The 5-ellipse is an algebraic curve of degree 32:



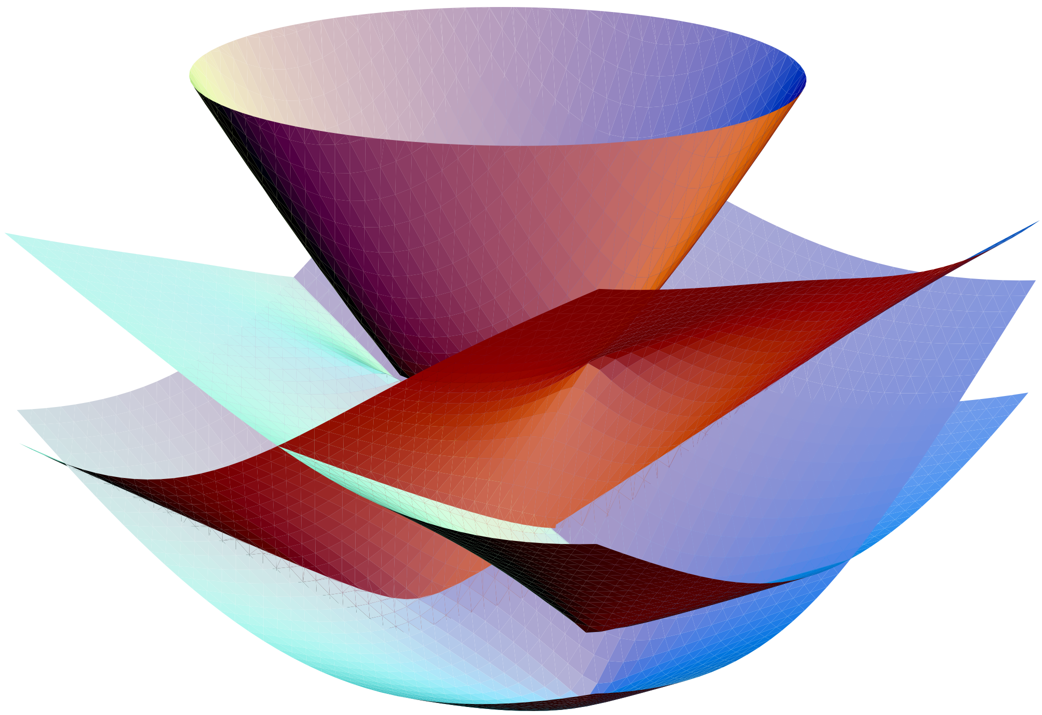
Concentric Ellipses

What is the algebraic degree of the m-ellipse?
How to write its equation?



What is the smallest radius d for which the m-ellipse is non-empty? How to compute the Fermat-Weber point?

3D View



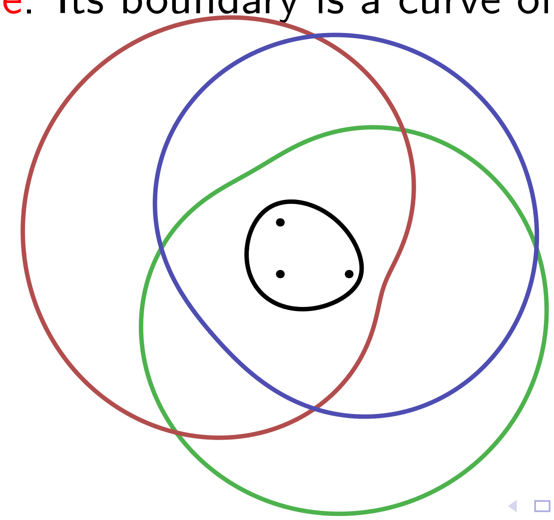
$$\mathcal{C} \;=\; \left\{ (x, y, d) \in \mathbb{R}^3 \;:\; \sum_{k=1}^{m} \sqrt{(x-u_k)^2 + (y-v_k)^2} \;\leq\; d \right\}.$$

Ellipses are Spectrahedra

The 3-ellipse with foci $(0,0),(1,0),(0,1)$ has the representation

$$\begin{bmatrix} d+3x-1 & y-1 & y & 0 & y & 0 & 0 & 0 \\ y-1 & d+x-1 & 0 & y & 0 & y & 0 & 0 \\ y & 0 & d+x+1 & y-1 & 0 & 0 & y & 0 \\ 0 & y & y-1 & d-x+1 & 0 & 0 & 0 & y \\ y & 0 & 0 & 0 & d+x-1 & y-1 & y & 0 \\ 0 & y & 0 & 0 & y-1 & d-x-1 & 0 & y \\ 0 & 0 & y & 0 & y & 0 & d-x+1 & y-1 \\ 0 & 0 & 0 & y & 0 & y & y-1 & d-3x+1 \end{bmatrix}$$

The ellipse consists of all points (x,y) where this 8×8-matrix is positive semidefinite. Its boundary is a curve of degree eight:



2, 2, 8, 10, 32, 44, 128, ...

Theorem: *The polynomial equation defining the m-ellipse has degree* 2^m *if m is odd* *and degree* $2^m - \binom{m}{m/2}$ *if m is even. We express this polynomial as the determinant of a symmetric matrix of linear polynomials. Our representation extends to weighted m-ellipses and m-ellipsoids in arbitrary dimensions .....*

[J. Nie, P. Parrilo, B.St.: Semidefinite representation of the k-ellipse, in *Algorithms in Algebraic Geometry*, I.M.A. Volumes in Mathematics and its Applications, 146, Springer, New York, 2008, pp. 117-132]

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CONVEX ALGEBRAIC GEOMETRY is the marriage of real algebraic geometry with optimization theory. It concerns convex figures such as ellipses, ellipsoids, . . . and much more.

The theorem says that m-ellipses and m-ellipsoids are *spectrahedra*.

Spectrahedra

A **spectrahedron** is the intersecton of the cone of positive semidefinite matrices with a linear space. **Semidefinite programming** is the computational problem of minimizing a linear function over a spectrahedron.

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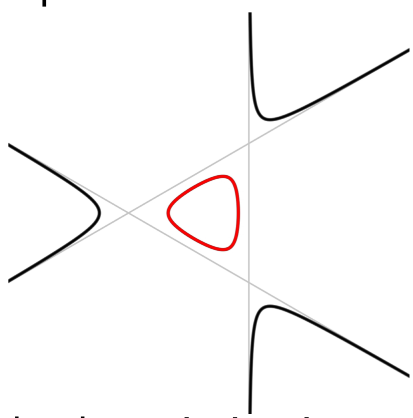
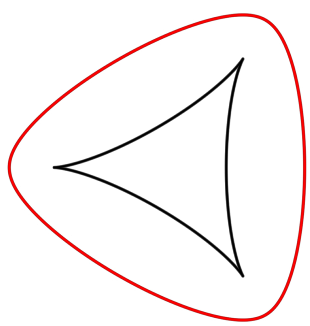
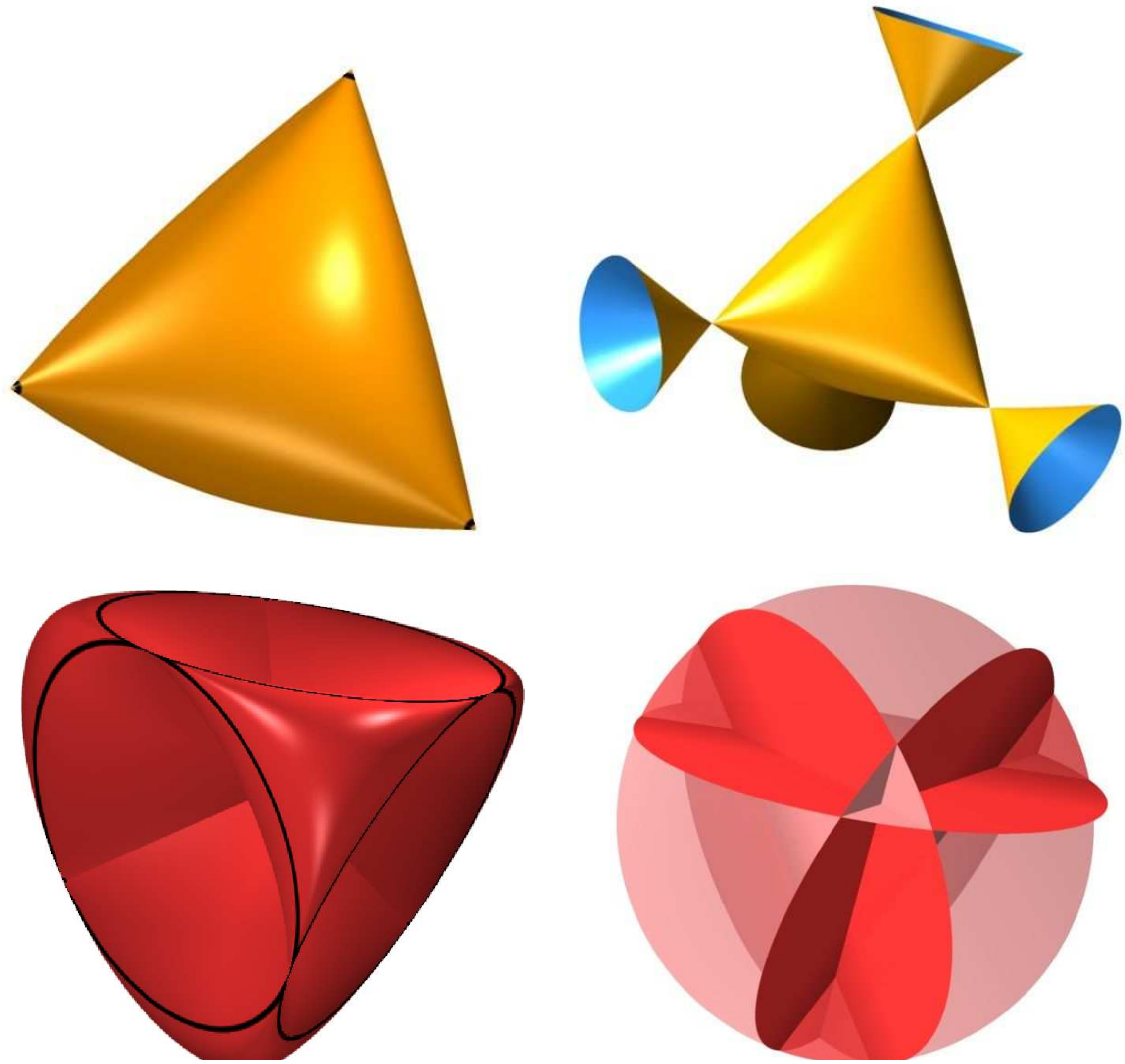


Duality is important in both optimization and projective geometry:



A Spectrahedron and its Dual



Minimizing Polynomial Functions

Let $f(x_1, \ldots, x_m)$ be a polynomial of even degree $2d$.
We wish to compute the global minimum x^* of $f(x)$ on $\mathbb{R}^m$.

This optimization problem is equivalent to

Maximize λ such that $f(x) - \lambda$ is non-negative on $\mathbb{R}^m$.

This problem is very hard.

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The optimal value of the following relaxtion gives a lower bound.

Maximize λ such that $f(x) - \lambda$ is a sum of squares of polynomials.

The second problem is much easier. It is a semidefinite program.

Empirically, the optimal value of the SDP often agrees with the global minimum. In that case, the optimal matrix of the dual SDP has rank one, and the optimal point x^* can be recovered from this.

SOS Programming: A Univariate Example

Let $m = 1$, $d = 2$ and $f(x) = 3x^4 + 4x^3 - 12x^2$. Then

$$f(x) - \lambda \quad = \quad \begin{pmatrix} x^2 & x & 1 \end{pmatrix} \begin{pmatrix} 3 & 2 & \mu - 6 \\ 2 & -2\mu & 0 \\ \mu - 6 & 0 & -\lambda \end{pmatrix} \begin{pmatrix} x^2 \\ x \\ 1 \end{pmatrix}$$

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Our problem is to find (λ, μ) such that the 3×3-matrix is positive semidefinite and λ is maximal. The optimal solution of this SDP is

$$(\lambda^*, \mu^*) \; = \; (-32, -2).$$

Cholesky factorization reveals the SOS representation

$$f(x) - \lambda^* \quad = \quad \left((\sqrt{3}\,x - \frac{4}{\sqrt{3}}) \cdot (x+2)\right)^2 + \frac{8}{3}(x+2)^2.$$

We see that the global minimum is $x^* = -2$.
This approach works for many polynomial optimization problems.