Math 255 Homework 5

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6.1 Let g be a holomorphic function on a Riemann surface S, and let $\gamma : [a,b] \to S$ be a closed piecewise-smooth path in S. Then by Example 6.14(2), $\int_\gamma dg = g(\gamma(b)) - g(\gamma(a)) = 0$ since $\gamma(b) = \gamma(a)$.

Let η be the holomorphic differential on $\mathbb{C}/\Lambda$ satisfying $\pi^*\eta = dz$. If we fix some nonzero $\lambda \in \Lambda$ and take the piecewise-smooth path $\gamma : [0,1] \to \mathbb{C}/\Lambda$ defined by $\gamma(t) = \Lambda + t\lambda$, then $\int_\gamma \eta = \lambda$, so η is not exact (see the discussion on page 150 for details).

6.3 Let dz be the meromorphic differential on $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$, where we identify $z \in \mathbb{C}$ with $[z, 1] \in \mathbb{P}^1$ and ∞ with $[1, 0]$. Let $\phi : \mathbb{P}^1 - \{\infty\} \to \mathbb{C}$, $\psi : \mathbb{P}^1 - \{0\} \to \mathbb{C}$ be holomorphic charts defined by $\phi[x, y] = x/y$, $\psi[x, y] = y/x$. To study the pole at ∞, we consider the chart ψ, with inverse $\psi^{-1}(z) = [1/z, 1]$ if $z \neq 0$, and $[1, 0]$ if $z = 0$. Since the meromorphic function $(1 \circ \psi^{-1})(z \circ \psi^{-1})' = -\frac{1}{z^2}$ has a pole of order 2 at $\psi(\infty) = 0$, hence dz has a pole of order 2 at ∞.

We can write any meromorphic differential hdg on $\mathbb{P}^1$ as fdz, where $f = hg'$ is meromorphic on $\mathbb{P}^1$. Suppose that fdz is holomorphic. Then on the chart ϕ, the function $(f \circ \phi^{-1})(z \circ \phi^{-1})' = f$ must be holomorphic. On the chart ψ, the function $(f \circ \psi^{-1})(z \circ \psi^{-1}) = f(\frac{1}{z})(-\frac{1}{z^2}) = -w^2 f(w)$ must be holomorphic, where $w = \frac{1}{z}$. So as $w \to \infty$, $w^2 f(w)$ must tend to a finite limit. Conversely, if these latter conditions hold then fdz has no poles and is a holomorphic differential.

If fdz is a holomorphic differential on $\mathbb{P}^1$, then from the above, f must have a zero of order at least 2 at ∞, and must also be constant, which is a contradiction. Hence there are no holomorphic differentials on $\mathbb{P}^1$.

6.5 Let C_Λ be the nonsingular cubic curve in $\mathbb{P}_2$ associated with a lattice Λ, defined by the polynomial $y^2 z - 4x^3 + g_2(\Lambda)xz^2 + g_3(\Lambda)z^3$. Let $p = [a_1, b_1, 1]$ and $q = [a_2, b_2, 1]$ be points of C_Λ, $a_1 \neq a_2$, and let L be the line through p, q. By solving a system of linear equations, we can write the defining equation of L as $(b_1 - b_2)x + (a_2 - a_1)y + (a_1 b_2 - a_2 b_1)z = 0$. Let $r = [a_3, b_3, 1]$ be the other point of intersection of L with C_Λ. To compute a_3, we set $z = 1$ and eliminate y from the equation for C_Λ using the equation for L, to get

$$\left(\frac{a_2 b_1 - a_1 b_2 + (b_2 - b_1)x}{a_2 - a_1} \right)^2 = 4x^3 - g_2(\Lambda)x - g_3(\Lambda)\,. \tag{1}$$

This is a cubic polynomial in x, and since the sum of the roots $a_1 + a_2 + a_3$ is the coefficient of x^2 divided by 4, we get

$$a_3 = \frac{1}{4}\left(\frac{b_1 - b_2}{a_1 - a_2} \right)^2 - (a_1 + a_2)\,. \tag{2}$$

Substituting this into the equation for L, we get

$$b_3 = \left(\frac{b_1 - b_2}{a_1 - a_2} \right) a_3 + \left(\frac{a_1 b_2 - a_2 b_1}{a_1 - a_2} \right)\,. \tag{3}$$

Let $z_1, z_2 \notin \Lambda$ and $z_1 \notin \Lambda \pm z_2$. Then since $z_1 + z_2 - (z_1 + z_2) \in \Lambda$, by Abel's theorem, there exists a line L in $\mathbb{P}_2$ intersecting C_Λ at $u(\Lambda + z_1)$, $u(\Lambda + z_2)$ and $u(\Lambda - (z_1 + z_2))$. Let $a_i = \wp(z_i)$, $b_i = \wp'(z_i)$ for

$i = 1, 2$ and let $a_3 = \wp(-(z_1 + z_2))$, $b_3 = \wp'(-(z_1 + z_2))$. Since $\wp$ is even, $a_3 = \wp(z_1 + z_2)$. Substituting these into Equation 2, we get

$$\wp(z_1 + z_2) = \frac{1}{4}\left(\frac{\wp'(z_1) - \wp'(z_2)}{\wp(z_1) - \wp(z_2)}\right)^2 - \wp(z_1) - \wp(z_2)\,. \tag{4}$$

If we let $z := z_1$, $z_2 = z + w$ and take the limit as $w \to 0$, then we get

$$\wp(2z) = \frac{1}{4}\left(\frac{\wp''(z)}{\wp'(z)}\right)^2 - 2\wp(z)\,. \tag{5}$$

6.6 Let $p \neq [0, 1, 0]$ be a point of the cubic curve C_Λ associated with a lattice Λ in $\mathbb{C}$. From Remark 6.22, p has order 2, or $p + p = 0$, if and only if the tangent to C_Λ at p passes through the identity, which is $[0, 1, 0]$. The points of order 1 or 2 correspond under the group isomorphism $u : \mathbb{C}/\Lambda \to C_\Lambda$ to points of order 1 or 2 in $\mathbb{C}/\Lambda$, and there are 4 such points which can be written in the form $\Lambda + \frac{j}{2}\omega_1 + \frac{k}{2}\omega_2$, for $j, k \in \{0, 1\}$. These points form a subgroup of C/Λ isomorphic to $C_2 \times C_2$.

6.7 Let n be a positive integer. If $p \in C_\Lambda$ has order dividing n, then $np = 0$, which by Abel's theorem occurs if and only if $nt \in \Lambda$ for some $t \in \mathbb{C}$ such that $u(\Lambda + t) = p$. So the points of order dividing n correspond to points of order dividing n in $\mathbb{C}/\Lambda$, and there are precisely n^2 such points which can be written in the form $\Lambda + \frac{j}{n}\omega_1 + \frac{k}{n}\omega_2$, where $j, k \in \{0, 1, \ldots, n-1\}$. These points form a subgroup of $\mathbb{C}/\Lambda$ isomorphic to the product of two cyclic groups of order n.

Let $q \in C_\Lambda$, such that q is not a point of inflection. Then the points in C_Λ whose tangent lines pass through q are the points p such that $p + p = -q$, so that $p + p + q = 0$. Let $t, v \in \mathbb{C}$ such that $u(\Lambda + t) = p$ and $u(\Lambda + v) = q$. Then by Abel's theorem, $p + p + q = 0$ if and only if $2t + v \in \Lambda$, or $t \in \frac{1}{2}\Lambda - \frac{1}{2}v$. So there are exactly four possibilities for t, which are $t = \frac{j}{2}\omega_1 + \frac{k}{2}\omega_2 - \frac{1}{2}v$, $j, k \in \{0, 1\}$, and these correspond to the four points in C_Λ other than q whose tangent lines pass through q.

6.8 Let $u, v, w \in \mathbb{C} - \Lambda$, such that u, v, w are distinct modulo Λ. By Abel's theorem, $u + v + w \in \Lambda$ if and only if there is a line L in $\mathbb{P}^2$ which intersects C_Λ at $[\wp(u), \wp'(u), 1]$, $[\wp(v), \wp'(v), 1]$,$[\wp(w), \wp'(w), 1]$, which happens if and only if

$$\det\begin{pmatrix} \wp(u) & \wp'(u) & 1 \\ \wp(v) & \wp'(v) & 1 \\ \wp(w) & \wp'(w) & 1 \end{pmatrix} = 0\,.$$