# Math 255 Homework 4

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5.4 Let $R$ be a compact connected Riemann surface. Suppose that there is a nonconstant holomorphic function $f : R \to \mathbb{C}$. Then $f$ extends to a holomorphic function to $\mathbb{P}^1$, which by Exercise 5.3 is surjective. Hence the image of $f$ must contain $\infty$, which is a contradiction.

5.9 Let $f : \mathbb{C} \to \mathbb{C}$ be a holomorphic doubly periodic function, with periods $\omega_1, \omega_2$, and let $\Lambda = \{n\omega_1 + m\omega_2 : n, m \in \mathbb{Z}\}$. By Example 5.42, $f$ corresponds to a holomorphic function $h : \mathbb{C}/\Lambda \to \mathbb{C}$. Since $\mathbb{C}/\Lambda$ is a compact connected Riemann surface, Exercise 5.4 implies that $h$ is constant. So $f$ must also be constant.

5.10 Let $\tilde{\wp} : \mathbb{C}/\Lambda \to \mathbb{P}^1$ be defined by $\tilde{\wp}(\Lambda + z) = \wp(z)$. Consider the holomorphic atlas on $\mathbb{P}^1$ given by the charts $\psi_1 : W_1 \to \mathbb{C}$, $\psi_2 : W_2 \to \mathbb{C}$, where $W_1 = \mathbb{P}^1 - \{\infty\}$ and $W_2 = \mathbb{P}^1 - \{0\}$, such that $\psi_1[x, y] = x/y$ and $\psi_2[x, y] = y/x$ (see Example 5.40(d)).

Let $\pi : \mathbb{C} \to \mathbb{C}/\Lambda$ be the map $z \mapsto \Lambda + z$. Consider the holomorphic charts on $\mathbb{C}/\Lambda$ given by $\phi_\alpha = (\pi|_{U_\alpha})^{-1} : \pi(U_\alpha) \to U_\alpha$, like in Example 5.42.

For $\Lambda + z \in \mathbb{C}/\Lambda$, take holomorphic charts $\phi_\alpha$, $\psi_i$ such that $\Lambda + z \in U_\alpha$ and $\tilde{\wp}(\Lambda + z) \in W_i$. To show that $\tilde{\wp}$ is holomorphic, we check that $\psi_i \circ \tilde{\wp} \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap \tilde{\wp}^{-1}(W_i)) \to \mathbb{C}$ is holomorphic. Observe that $\tilde{\wp} \circ \phi_\alpha^{-1} = \tilde{\wp} \circ \pi = \wp$, and $U_\alpha \cap \tilde{\wp}^{-1}(W_i)$ is the set of points in $U_\alpha$ which are not in $\Lambda$, so $\wp$ is holomorphic on $\pi^{-1}(U_\alpha \cap \tilde{\wp}^{-1}(W_i)) = \phi_\alpha(U_\alpha \cap \tilde{\wp}^{-1}(W_i))$, hence $\psi_i \circ \tilde{\wp} \circ \phi_\alpha^{-1}$ is holomorphic. Thus $\tilde{\wp}$ is holomorphic.

Let $f(z) = (\wp(z) - \wp(\frac{1}{2}\omega_1))(\wp(z) - \wp(\frac{1}{2}\omega_2))(\wp(z) - \wp(\frac{1}{2}\omega_1 + \frac{1}{2}\omega_2))$ and let $g(z) = f(z)/\wp'(z)^2$. Since $\wp, \wp'$ have poles of order 2 and 3 respectively at every $z \in \Lambda$, and no other poles, hence $g$ is holomorphic at each $z \in \Lambda$. For $z \in \frac{1}{2}\Lambda - \Lambda$, by Lemma 5.13, $\wp(z) = \wp(\frac{1}{2}\omega_1)$ or $\wp(\frac{1}{2}\omega_2)$ or $\wp(\frac{1}{2}\omega_1 + \frac{1}{2}\omega_2)$. So $f$ has zeros at each $z \in \frac{1}{2}\Lambda - \Lambda$, and these zeros have order 2. Since $\wp'$ has a simple zero at each of these points, hence $g$ is holomorphic at each $z \in \frac{1}{2}\Lambda - \Lambda$. Thus $g$ is holomorphic on $\mathbb{C}$, since $\wp'$ has no other zeros. $g$ is also doubly periodic since $\wp, \wp'$ are doubly periodic.

By Exercise 5.9, $g$ is constant, so $g(z) = c$ for some constant $c$. Thus $\wp'(z)^2 = \frac{1}{c}f(z) = Q(\wp(z))$, where $Q(x) = \frac{1}{c}(x - \wp(\frac{1}{2}\omega_1))(x - \wp(\frac{1}{2}\omega_2))(x - \wp(\frac{1}{2}\omega_1 + \frac{1}{2}\omega_2))$ is a cubic polynomial.

5.12 Consider the polynomial $f = 4x^3 - g_2(\Lambda)x - g_3(\Lambda)$. By the proof of Lemma 5.20, $f$ has distinct roots. Hence the discriminant of $f$, which is $g_2(\Lambda)^3 - 27g_3(\Lambda)^2$, is nonzero.

5.14 Suppose that there is a projective transformation of $\mathbb{P}^2$ given by a diagonal matrix taking $C$ to $\tilde{C}$, then this transformation is of the form $x \mapsto ax$, $y \mapsto by$, $z \mapsto cz$ for $a, b, c \neq 0$. We can assume $c = 1$ since this is a projective transformation. Substituting this into the equation for $C$, we get $b^2y^2z = 4a^3x^3 - g_2axz^2 - g_3z^3$. Comparing this with the equation for $\tilde{C}$, we get $a^3 = b^2$, $g_2a = b^2\tilde{g}_2$ and $g_3 = b^2\tilde{g}_3$. We reparameterize this by letting $a = u^2$, then $b = u^3$, and $g_2 = u^4\tilde{g}_2$, $g_3 = u^6\tilde{g}_3$. So $J(C) = \frac{u^{12}\tilde{g}_2}{u^{12}\tilde{g}_2 - 27 * u^{12}\tilde{g}_3} = J(\tilde{C})$.

Conversely, suppose $J(C) = J(\tilde{C})$. Then $g_2^3\tilde{g}_3^2 = \tilde{g}_2^2 g_3^2$, so for some nonzero $u$ we can write $(g_3/\tilde{g}_3)^2 = (g_2/\tilde{g}_2)^3 = u^{12}$. Consider a projective transformation of the form $x \mapsto u^2x$, $y \mapsto u^3y$; this is given by a diagonal matrix. Then the equation of $C$ is mapped to $y^2z = 4x - (g_2/u^4)xz^2 - (g_3/u^6)z^3 = 4x - \tilde{g}_2xz^2 - \tilde{g}_3z^3$, so this transformation takes $C$ to $\tilde{C}$.

5.18 (i) Since $C$ is nonsingular, there is no point $[x, y, z] \in C$ such that $\frac{\partial P}{\partial x} = \frac{\partial P}{\partial y} = \frac{\partial P}{\partial z} = 0$, so the image of $C$ is in $\mathbb{P}^2$ and is well-defined. Euler's relation implies that the points in the image satisfy $x\frac{\partial P}{\partial x} + y\frac{\partial P}{\partial y} + z\frac{\partial P}{\partial z} = 0$, so the image is defined by a homogeneous polynomial and is a projective curve.

(ii) Let $C$ be a conic, with defining equation $P(x, y, z) = a_1x^2 + a_2xy + a_3xz + a_4y^2 + a_5yz + a_6z^2 = 0$. Then $\frac{\partial P}{\partial x} = 2a_1x + a_2y + a_3z$, $\frac{\partial P}{\partial y} = a_2x + 2a_4y + a_5z$, $\frac{\partial P}{\partial z} = a_3x + a_5y + 2a_6z$, so the polar mapping is linear and is a projective transformation. Hence the dual curve is also a nonsingular conic.

(iii) The polar mapping from $C$ to $\tilde{C}$ is defined by polynomials and so is holomorphic. Suppose the degree of $C$ is at least 3. By Proposition 3.33(ii), $C$ has at least one point of inflection. Since points of inflection correspond to cusps on the dual curve, hence $\tilde{C}$ has a cusp, and so there is no holomorphic inverse.