Math 255 Homework 2

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3.3 A conic is a curve of the form $a_1x^2 + a_2xy + a_3xz + a_4y^2 + a_5yz + a_6z^2$, and has 6 coefficients, thus the space of conics is 5-dimensional. Given 5 points in $\mathbb{P}^2$, each point specifies a linear constraint on the coefficients, and since there are 5 dimensions, we can solve the system of equations to find a conic containing the 5 points.

Let C be a projective curve of degree 4 in $\mathbb{P}^2$ with four singular points. Let D be a conic containing the 4 singular points and another point of C. By Bézout's theorem, if the two curves have no common component, then they have 8 points of intersection counting multiplicities. Since the 4 singular points have multiplicity greater than 1, and C and D have an additional point of intersection, hence the sum of the intersection multiplicities is at least 9. Thus C and D must have a common component, which implies C is reducible.

3.8 Let C be a projective curve in $\mathbb{P}^2$ defined by a homogeneous polynomial P, let α be a linear transformation, and let $Q = P \circ \alpha^{-1}$. We label the coordinates on P by x_1, x_2, x_3 and the coordinates on Q by v_1, v_2, v_3, such that $\alpha^{-1}(v_1, v_2, v_3) = (x_1, x_2, x_3)$. To compute the derivative of Q at some $v \in \mathbb{C}^3 - \{0\}$, we use the chain rule to get $\frac{\partial Q}{\partial v_i} = \frac{\partial P}{\partial x_1}\frac{\partial x_1}{\partial v_i} + \frac{\partial P}{\partial x_2}\frac{\partial x_2}{\partial v_i} + \frac{\partial P}{\partial x_3}\frac{\partial x_3}{\partial v_i}$, for $i = 1, 2, 3$. Since α^{-1} is a linear transformation, it is given by some matrix, and $\frac{\partial x_i}{\partial v_j}$ is the (i, j)-th entry of the matrix. So we get that the gradient of Q evaluated at v (as a column vector), is equal to the gradient of P evaluated at $\alpha^{-1}(v)$, multiplied on the left by the transpose of the matrix of α^{-1}.

Applying the chain rule again, we get that the matrix of second derivatives of Q at v is the matrix of second derivatives of P at $\alpha^{-1}(v)$, multiplied on the right and the left by the matrix of α^{-1} and its transpose respectively. By taking determinants, we get the desired identity on the Hessians.

This implies that if the Hessian of a polynomial vanishes at a point, it also vanishes after a linear transformation, so the definition of an inflection point is invariant under projective transformations.

3.11 By Corollary 3.34, there is a projective transformation taking p to $[0, 1, 0]$ and taking C to a curve of the form $y^2z = x(x - z)(x - \lambda z)$, where $\lambda \in \mathbb{C}$, $\lambda \neq 0, 1$. We compose this with the transformation $x \mapsto \frac{\lambda+1}{3}z, y \mapsto \frac{1}{2}y$, to get the equation

$$y^2z = 4x^3 - \frac{4}{3}(\lambda^2 - \lambda + 1)xz^2 - 4\left(\frac{2}{27}\lambda^3 - \frac{1}{9}\lambda^2 - \frac{1}{9}\lambda + \frac{2}{27}\right)z^3 \tag{1}$$

So we have $g_2 = \frac{4}{3}(\lambda^2 - \lambda + 1)$ and $g_3 = 4(\frac{2}{27}\lambda^3 - \frac{1}{9}\lambda^2 - \frac{1}{9}\lambda + \frac{2}{27})$. Then $g_2^3 - 27(g_3)^2 = 16\lambda^2(\lambda - 1)^2 \neq 0$ since $\lambda \neq 0, 1$.

3.13 Suppose that C and D meet in exactly 9 points $p_1, \ldots, p_9$. Suppose that some line L in $\mathbb{P}^2$ contains 4 of these points. By Bézout's theorem, a line and a cubic and only meet in 3 points unless they have a common component, thus C must contain L as a component. Applying the same argument to D, this implies that C and D both contain L, and so cannot meet in exactly 9 points. Hence this is a contradiction, and no line in $\mathbb{P}^2$ can contain four of the points.

Similarly, a conic can only meet a cubic in 6 points unless they have a common component. So if a conic meets C in 7 points, it must have a common component with C, and similarly for D. Moreover, this must be the same component, so C and D cannot meet in exactly 9 points. Hence no conic contains seven of the points.

From exercise 3.3, there is at least one conic containing $p_1, \dots, p_5$. Moreover, since no line contains four of these, this conic Q is unique.

Suppose that E contains $p_1, \dots, p_8$ and that R is not a linear combination of P and Q. Let q, r be distinct points in $\mathbb{P}^2$. Then we can find a curve C defined by $\lambda P + \mu Q + \nu R = 0$, with $\lambda, \mu, \nu \in \mathbb{C}$, which passes through $p_1, \dots, p_8, q, r$. We do this by substituting q, r into the equation to get two linear equations in λ, μ, ν, which we can solve for a solution with nonzero λ, μ, ν.

Suppose that p_8 lies in the line L through p_6, p_7, and choose $q \in L$, $r \notin L \cup Q$. Then C contains the four points $p_6, p_7, p_8, q \in L$, so by Bézout's theorem, C contains L as a component. So C is the union of L and a conic, which must be Q since Q is the unique conic containing $p_1, \dots, p_5$. This is a contradiction since $r \notin L \cup Q$, so p_6, p_7, p_8 cannot lie on a line. Applying the same argument to the other points, we deduce that no three of $p_1, \dots, p_8$ lie on a line.

If $p_8 \notin Q$, and $q, r \in L$, then by the same argument we can show that $C = L \cup Q$, which is a contradiction. Similarly, we get a contradiction assuming $p_6 \notin Q$, or $p_7 \notin Q$. Thus we deduce that $p_6, p_7, p_8 \in Q$. This implies that Q contains 8 points of the points, which contradicts our proof earlier that no conic contains seven of the points. So the original hypotheses on E were inconsistent. Hence if E contains $p_1, \dots, p_8$, then R must be a linear combination of P and Q, such that E also contains p_9.

3.14 Since $D = L_1 \cup M_2 \cup L_3$, D meets C in the points where each of the lines in D meets C, which are $p, q, -(p+q)$ for L_1, $p_0, q+r, -(q+r)$ for M_2 and $r, p+q, -((p+q)+r)$ for L_3. So D meets C in the points $p_0, p, q, r, p+q, q+r, -(p+q), -(q+r), -((p+q)+r)$, which we label as $p_1, \dots, p_9$ respectively. Similarly, we can check that E meets C in the points $p_0, p, q, r, p+q, q+r, -(p+q), -(q+r), -(p+(q+r))$, which are $p_1, \dots, p_8$ and a ninth point which we label p_9'. We apply the result from Exercise 3.13 as follows. Since C and D meet in exactly the nine points $p_1, \dots, p_9$, and E contains $p_1, \dots, p_8$, by Exercise 3.13, E also contains p_9. Moreover, since E meets C in exactly nine points, hence $p_9' = p_9$, which implies $(p+q)+r = p+(q+r)$.

3.16 Let p be a point of inflection of a nonsingular cubic curve C in $\mathbb{P}^2$. Then by Remark 3.35, there is a projective transformation taking p to $[0,1,0]$ and taking C to a curve $y^2z = x(x-z)(x-\lambda z)$ for some $\lambda \in \mathbb{C} - \{0,1\}$. Let $f = y^2z - x(x-z)(x-\lambda z)$. To compute the tangent lines to C which pass through p, we first solve for the points $[a,b,c]$ where $\frac{\partial f}{\partial x}(a,b,c)\cdot 0 + \frac{\partial f}{\partial y}(a,b,c)\cdot 1 + \frac{\partial f}{\partial z}(a,b,c)\cdot 0 = \frac{\partial f}{\partial y}(a,b,c) = 0$. This implies $2yz = 0$. If $y = 0$, then $x = 0$, $x = z$ or $x = \lambda z$, which gives us the points $[0,0,1], [1,0,1], [\lambda,0,1]$ respectively. If $z = 0$, then $x = 0$ and we have the point $[0,1,0]$. Hence there are exactly four tangent lines to C which pass through p, which are the four tangent lines to C at these points.