Math 255 Homework 1

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2.2 (a) Let $f = y^3 - y^2 + x^3 - x^2 + 3y^2x + 3x^2y + 2xy$, then

$$\frac{\partial f}{\partial x} = 3x^2 - 2x + 3y^2 + 6xy + 2y$$
$$\frac{\partial f}{\partial y} = 3y^2 - 2y + 6xy + 3x^2 + 2y$$

Subtracting the bottom equation from the top one, we get $y = x$, which implies that the only solution is $x = 0, y = 0$. So the only singular point is $(0,0)$, with multiplicity 2. To find the tangent lines, we factorize the polynomial from (2.1), which is $-x^2 - y^2 + 2xy = -(x-y)^2$. So the tangent line is $y = x$, and the multiplicity is not ordinary.

(b) Let $f = x^4 + y^4 - x^2y^2$, then

$$\frac{\partial f}{\partial x} = 4x^3 - 2xy^2 = 2x(\sqrt{2}x + y)(\sqrt{2}x - y) = 0$$
$$\frac{\partial f}{\partial y} = 4y^3 - 2x^2y = 2y(\sqrt{2}y + x)(\sqrt{2}y - x) = 0$$

The only solution is $x = 0, y = 0$, so the only singular point is $(0,0)$, with multiplicity 4. To find the tangent lines, we factorize the polynomial $x^4 - x^2y^2 + y^4$, to get

$$\begin{aligned}
&x^4 - x^2y^2 + y^4 \\
&= \left(x^2 - \frac{1+i\sqrt{3}}{2}y^2\right)\left(x^2 - \frac{1-i\sqrt{3}}{2}y^2\right) \\
&= \left(x + \sqrt{\frac{1+i\sqrt{3}}{2}}y\right)\left(x - \sqrt{\frac{1+i\sqrt{3}}{2}}y\right)\left(x + \sqrt{\frac{1-i\sqrt{3}}{2}}y\right)\left(x - \sqrt{\frac{1-i\sqrt{3}}{2}}y\right)
\end{aligned}$$

This gives us the 4 tangent lines $x = -\sqrt{\frac{1+i\sqrt{3}}{2}}y$, $x = \sqrt{\frac{1+i\sqrt{3}}{2}}y$, $x = -\sqrt{\frac{1-i\sqrt{3}}{2}}y$, $x = \sqrt{\frac{1-i\sqrt{3}}{2}}y$.

(c) Let $f = y^2 - x^3 + x$, then $\frac{\partial f}{\partial x} = -3x^2 + 1$ and $\frac{\partial f}{\partial y} = 2y$. The solutions to these equations are $(\frac{1}{\sqrt{3}}, 0)$ and $(-\frac{1}{\sqrt{3}}, 0)$, which are not on the curve. Hence the curve has no singular point.

2.4 The point (a,b) is an ordinary double point if and only if the polynomial from (2.1) is not a square. We can write the polynomial as

$$\begin{aligned}
&\frac{\partial^2 P}{\partial x \partial y}(x-a)(y-b) + \frac{1}{2}\frac{\partial^2 P}{\partial x^2}(x-a)^2 + \frac{1}{2}\frac{\partial^2 P}{\partial y^2}(y-b)^2 \\
&= \frac{1}{2}\left(\sqrt{\frac{\partial^2 P}{\partial x^2}}(x-a) + \sqrt{\frac{\partial^2 P}{\partial y^2}}(y-b)\right)^2 + \left(\frac{\partial^2 P}{\partial x \partial y} - \sqrt{\frac{\partial^2 P}{\partial x^2}\frac{\partial^2 P}{\partial y^2}}\right)(x-a)(y-b),
\end{aligned}$$

where the partial derivatives are all evaluated at (a,b). This is not a square if and only if

$$\left(\frac{\partial^2 P}{\partial x \partial y}\right)^2 \neq \frac{\partial^2 P}{\partial x^2}\frac{\partial^2 P}{\partial y^2}.$$

2.5 Let C be an affine curve defined by a polynomial $P(x,y)$ of degree d, and let (a,b) be a point of multiplicity d in C. We first assume by doing a change of coordinates that the point (a,b) is $(0,0)$. Since the multiplicity is d, all the partial derivatives of order less than d vanish, which implies that P is a homogeneous polynomial of degree d in x, y. Thus by Lemma 2.8, P factors as a product of d linear factors in $(x-a), (y-b)$, and so is the union of d lines through (a,b).

2.7 Let C be a complex algebraic curve in $\mathbb{C}^2$ defined by a nonconstant polynomial $P(x,y)$ with complex coefficients. Then for all but at most finitely many values of $a \in \mathbb{C}$, $P(a,y)$ is a nonconstant polynomial in y, which has at least one root $b \in \mathbb{C}$ such that $P(a,b)=0$. So C is not compact.

2.8 (a) Let $P = xy^4 + yz^4 + xz^4$. Then

$$\begin{aligned}\frac{\partial P}{\partial x} &= y^4 + z^4 = 0\,,\\ \frac{\partial P}{\partial y} &= 4xy^3 + z^4 = 0\,,\\ \frac{\partial P}{\partial z} &= 4yz^3 + 4xz^3 = 0\,.\end{aligned}$$

From the last equation, we have $z^3(y+x)=0$, so $z=0$ or $y=-x$. If $z=0$, then the first equation gives $y=0$, so the only solution is $[1,0,0]$. Suppose $y=-x$. Combining the first two equations gives $4xy^3 = y^4$, so $y=0$ or $4x=y$. If $y=0$, then $z=0$ from the first equation, and $x=-y=0$, which is not possible. If $4x=y$, then $4x=y=-x$, so again $x=y=z=0$. Thus the only singular point is $[1,0,0]$, with multiplicity 4.

(b) Let $P = x^2y^3 + x^2z^3 + y^2z^3$. Then

$$\begin{aligned}\frac{\partial P}{\partial x} &= 2xy^3 + 2xz^3 = 2x(y^3+z^3) = 0\,,\\ \frac{\partial P}{\partial y} &= 3x^2y^2 + 2yz^3 = y(3x^2y + 2z^3) = 0\,,\\ \frac{\partial P}{\partial z} &= 3x^2z^2 + 3y^2z^2 = 3z^2(x^2+y^2) = 0\,.\end{aligned}$$

If $x=0$, then either $y=0$ or $z=0$, and we get the singular points $[0,0,1]$, $[0,1,0]$. Similarly, by considering the cases where $y=0$ or $z=0$, we get the singular point $[1,0,0]$.

Suppose $y,z \neq 0$. Then $x^2+y^2=0$ and $3x^2y+2z^3 = -3y^3-2y^3 = -5y^3 = 0$, which is a contradiction. If $x,y \neq 0$, then $y^3+z^3=0$ and $3x^2y+2z^3 = 3x^2y - 2y^3 = 0$, so $3x^2 = 2y^2$. Substituting this into the third equation, we get $z=0$, which implies $y=0$ and is a contradiction. Similarly, if $x,z \neq 0$, then $y^3+z^3=0$ and $x^2+y^2=0$. Substituting these into the second equation, we get $y=0$, so $x=z=0$ and we have a contradiction.

Thus the singular points are $[1,0,0], [0,1,0], [0,0,1]$, with multiplicities $3,2,2$ respectively.

(c) Let $P = y^2z - x(x-z)(x-\lambda z) = y^2z - x^3 + \lambda x^2 z + x^2 z - \lambda x z^2$. Then

$$\begin{aligned}\frac{\partial P}{\partial x} &= -3x^2 + 2\lambda xz + 2xz - \lambda z^2\,,\\ \frac{\partial P}{\partial y} &= 2yz\,,\\ \frac{\partial P}{\partial z} &= y^2 + \lambda x^2 + x^2 - 2\lambda xz\,.\end{aligned}$$

From the second equation, we have $y=0$ or $z=0$. If $z=0$, then $x=0$ and $y=0$, which is not possible. If $y=0$, we get from the last equation that $x(\lambda x + x - 2\lambda z)=0$. Then either $x=0$, which can only happen if $\lambda=0$, or $(\lambda+1)x = 2\lambda z$. This gives us the singular point $[2\lambda, 0, \lambda+1]$ of multiplicity 2.

(d) Let $P = x^n + y^n + z^n$, then $\frac{\partial P}{\partial x} = nx^{n-1}$, $\frac{\partial P}{\partial y} = ny^{n-1}$, $\frac{\partial P}{\partial z} = nz^{n-1}$, which has no solution in $\mathbb{P}^2$. Thus there are no singular points.

3.1 Let C, D be projective curves in $\mathbb{P}^2$ with no common component, defined by the polynomials $P(x, y, z)$ and $Q(x, y, z)$ respectively. Then the singular points of $C \cup D$ are defined by the vanishing of the first partial derivatives of PQ with respect to x, y, z. Since $\frac{\partial}{\partial x}(PQ) = P\frac{\partial}{\partial x}Q + Q\frac{\partial}{\partial x}P$, this vanishes on $C \cap D$. Moreover, it also vanishes on the singularities of C, since these are the points where $P = \frac{\partial}{\partial x}P = 0$, and similarly it vanishes on the singularities of D. The same holds for the partial derivatives with respect to y, z. So the singularities of $C \cup D$ are the singularities of C, D and the points in $C \cap D$.

By Corollary 3.10, an irreducible projective curve in $\mathbb{P}^2$ has at most finitely many singular points. Any projective curve C in $\mathbb{P}^2$ defined by a polynomial with no repeated factors is a union of irreducible curves, and the singularities of C would be the union of the singularities of each of the irreducible curves and the points of their intersection. Since each irreducible curve has finitely many singular points, and they intersect in finitely many points, hence C has finitely many singular points.

3.6 Consider the two projective curves of degree three defined by $L_{21}L_{32}L_{13}$ and $L_{12}L_{23}L_{31}$, and let the defining polynomials be P and Q respectively. They intersect in the points $p_1, p_2, p_3, q_1, q_2, q_3$, as well as the three points of intersection of the pairs of lines L_{ij} and L_{ji}, which we will call r_1, r_2, r_3.

Let $[a, b, c]$ be the point of intersection of L and M, and take the curve of degree 3 defined by $\lambda P + \mu Q$, where $\lambda = Q(a, b, c)$ and $\mu = -P(a, b, c)$. This curve meets $L \cup M$ in at least 7 points, so by Bézout's theorem, they must have a common component, which must be one of the lines L or M. Without loss of generality, we assume that it is L. Let R_1, R_2 be the defining polynomials of L, M respectively. Then $\lambda P + \mu Q = R_1 S$ for some polynomial S of degree 2, where S vanishes on $q_1, q_2, q_3, r_1, r_2, r_3$.

We apply Bézout's theorem again as follows. Since the curve defined by S has degree 2 and meets M in at least 3 points, they must have a common component, which must be M since it is irreducible. So S is a product of two lines, defined by R_2 and some other polynomial R_3. Then the 3 points r_1, r_2, r_3, which do not lie on L and M, must lie on the line defined by R_3.