1. [Fulton 2.26] Fix a DVR $R$ with quotient field $K$ and maximal ideal $m$.
   (a) Show that if $z \in K \setminus R$ then $z^{-1} \in m$.
   (b) Suppose $R \subseteq S \subset K$, where $S$ is a DVR whose maximal ideal contains $m$. Prove that $R = S$.

2. [Fulton 2.44] Let $V$ be a variety in $\mathbb{A}^n$, $I$ its ideal in $k[X_1, \ldots, X_n]$, and $P \in V$. Prove that $\mathcal{O}_P(V)$ is isomorphic to $\mathcal{O}_P(\mathbb{A}^n)/I\mathcal{O}_P(\mathbb{A}^n)$. How would you perform computations in this ring using Macaulay2?

3. [Fulton 4.27] Show that the pole set of a rational function on a variety in any multispace $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r} \times \mathbb{A}^m$ is an algebraic subset.

4. Fix points $P_1, P_2, \ldots, P_n$ in $\mathbb{P}_k^2$ and consider the function
   \[ \Phi : N^{n+1} \to N, \ (d, r_1, \ldots, r_n) \mapsto \dim_k(V(d, r_1 P_1, \ldots, r_n P_n)) \]
   True or false: The function $\Phi$ is piecewise polynomial.

5. [Fulton 6.45] Let $C$ and $C'$ be curves and $f$ a rational map from $C$ to $C'$.
   (a) Prove that $f$ is either dominating or constant.
   (b) Show: if $f$ is dominating then $k(C')$ is a finite extension of $k(C)$.
6. [Fulton 7.9] Draw the set of real points of the quartic curve
\[ C = \mathcal{V}(X^4 + Y^4 - XY^2) \subset \mathbb{P}^2. \]
Write down equations for a nonsingular curve \( X \) in some \( \mathbb{P}^N \) that is birationally equivalent to \( C \)? Does there exist a computer program that can perform both of these two tasks for arbitrary plane curves?

7. [Fulton 8.11] Let \( D \) be a divisor on an irreducible projective curve. Show that \( l(D) > 0 \) if and only if \( D \) is linearly equivalent to an effective divisor.

8. **Bonus Problem:** The \( k \)-ellipse is the Zariski closure in \( \mathbb{P}^2_\mathbb{C} \) of the curve consisting of all points in \( \mathbb{R}^2 \) that have a fixed distance from \( k \) given generic points. Determine the singularities and the genus of the \( k \)-ellipse, i.e. answer the Open Question #1 in §5 of arXiv/0702005.