NOTES ON HYPERBOLICITY CONES

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1. Hyperbolic programming

A *hyperbolic program* is an optimization problem of the form

$$\begin{aligned} &\text{minimize } c^T x \\ &\text{such that } Ax = b \text{ and} \\ &x \in \Lambda_+, \end{aligned}$$

where $c \in \mathbb{R}^n$, $Ax = b$ is a system of linear equations and Λ_+ is the closure of a so called hyperbolicity cone. Hyperbolic programming generalizes semidefinite programming, but it is not known to what extent since it is not known how general the hyperbolicity cones are. The rich algebraic structure of hyperbolicity cones makes hyperbolic programming an interesting context for optimization. For further reading we refer to [2, 7] and the references therein.

2. Stable and hyperbolic polynomials

Stable and hyperbolic polynomials are both generalizations of univariate polynomials with only real zeros.

Let $\mathbb{H} = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$. A polynomial $p(x) \in \mathbb{C}[x_1, \dots, x_n]$ is *stable* if

$$z \in \mathbb{H}^n \quad \Longrightarrow \quad p(z) \neq 0.$$

Here are two elementary examples:

(1) Suppose that $p(x) \in \mathbb{R}[x]$. Then $p(x)$ is stable if and only if it has only real zeros (since non-real zeros come in conjugate pairs).

(2) Let $p(x, y) = \sum_{k=0}^d a_k x^k y^{d-k} \in \mathbb{R}[x, y]$, be a homogenous polynomial of degree d with $a_d \neq 0$. Then p is stable if and only if $p(x, 1)$ has only real and non-positive zeros. Indeed if $p(x, 1)$ has only real and non-positive zeros, then we may write $p(x, y) = a_d \prod_{j=1}^d (x + \alpha_j y)$, where $\alpha_j \geq 0$ for all j. Since each term $x + \alpha_j y$ is stable, and since stability is closed under multiplication it follows that $p(x, y)$ is stable.

   On the other hand if $p(x, y)$ is stable, then so is $p(x, 1)$ by (3) below. Hence we may write $p(x, y) = a_d \prod_{j=1}^d (x + \alpha_j y)$, where $\alpha_j \in \mathbb{R}$ for all j. If $\alpha_j < 0$ for some j, then $p(x, y) = 0$ for $(x, y) = (|\alpha_j| i, i) \in \mathbb{H}^2$. Hence $\alpha_j \geq 0$ for all j and $p(x, y)$ is of the desired form.

(3) Let $\overline{\mathbb{H}}$ be the closed upper half plane of $\mathbb{C}$. If $p(x_1, \dots, x_n)$ is stable and $\eta \in \overline{\mathbb{H}}$, then $q(x_1, \dots, x_{n-1}) = p(x_1, \dots, x_{n-1}, \eta)$ is stable or identically zero. Indeed if $\epsilon > 0$ then $p(x_1, \dots, x_{n-1}, \eta + \epsilon i)$ is stable. Hence by letting $\epsilon \to 0$, and invoking Hurwitz' theorem on the continuity of zeros we see that $q(x_1, \dots, x_{n-1})$ is stable or identically zero.

Stable polynomials appear in complex analysis, control theory, statistical mechanics, probability theory and combinatorics. For a recent survey on new developments on stable polynomials see [8].

A homogeneous polynomial $h(x) \in \mathbb{R}[x_1, \ldots, x_n]$ is *hyperbolic* with respect to a vector $e \in \mathbb{R}^n$ if $h(e) \neq 0$, and if for all $x \in \mathbb{R}^n$ the univariate polynomial $t \mapsto h(x + et)$ has only real zeros. Hyperbolic polynomials have their origin in PDE theory where they were studied by Petrovsky, Gårding, Bott, Atiyah and Hörmander. During recent years hyperbolic polynomials have been studied in diverse areas such as control theory, optimization, probability theory computer science and combinatorics. Here are some examples of hyperbolic polynomials:

(1) Let $h(x) = x_1 \cdots x_n$. Then $h(x)$ is hyperbolic with respect to any vector $e \in \mathbb{R}^n$ that has no coordinate equal to zero:
$$h(x + et) = \prod_{j=1}^{n} (x_j + e_j t).$$
(2) Let $X = (x_{ij})_{i,j=1}^n$ be a matrix of variables where we impose $x_{ij} = x_{ji}$. Then $\det(X)$ is hyperbolic with respect to $I = \mathrm{diag}(1, \ldots, 1)$. Indeed $t \mapsto \det(X + tI)$ is the characteristic polynomial of the symmetric matrix X, so it has only real zeros.
(3) Let $h(x) = x_1^2 - x_2^2 - \cdots - x_n^2$. Then h is hyperbolic with respect to $(1, 0, \ldots, 0)^T$.

3. The hyperbolicity cone

Suppose that h is hyperbolic with respect to e, and of degree d. We may write
$$h(x + et) = h(e) \prod_{j=1}^{d} (t + \lambda_j(x)),$$
where $\lambda_1(x) \leq \cdots \leq \lambda_d(x)$. The *hyperbolicity cone* is the set
$$\Lambda_{++} = \Lambda_{++}(e) = \{x \in \mathbb{R}^n : \lambda_1(x) > 0\}.$$
Since $h(e + te) = h(e)(1 + t)^d$ we see that $e \in \Lambda_{++}$. The hyperbolicity cones for the examples given is Section 2 are:

(1) $\Lambda_{++}(e) = \{x \in \mathbb{R}^n : x_i e_i > 0 \text{ for all } i\}$.
(2) $\Lambda_{++}(I)$ is the cone of symmetric positive definite matrices.
(3) $\Lambda_{++}(1, 0, \ldots, 0)$ is the *Lorentz cone*
$$\{x \in \mathbb{R}^n : x_1 > \sqrt{x_2^2 + \cdots + x_n^2}\}.$$

Proposition 1. *The hyperbolicity cone is the connected component of*
$$\{x \in \mathbb{R}^n : h(x) \neq 0\}$$
which contains e.

Proof. Let C be the connected component that contains e. Suppose that $x(s)$, $0 \leq s \leq 1$ is a continous path in C connecting $e = x(0)$ and $x = x(1)$. Then $\lambda_1(x(s)) > 0$ for all $0 \leq s \leq 1$ for otherwise $\lambda_1(x(s)) = 0$ for some $0 \leq s \leq 1$ which implies $h(x(s)) = 0$ contrary to the assumption that $x(s) \in C$.

One the other hand if $x \in \Lambda_{++}$, then by homogeneity
$$h(tx + (1 - t)e) = h(e) \prod_{j=1}^{d} (t\lambda_j(x) + (1 - t)).$$
Since $\lambda_j(x) > 0$ for all j we see that $tx + (1 - t)e \in C$ for all $0 \leq t \leq 1$. $\square$

Lemma 2. *Let $h(x)$ be a homogeneous polynomial of degree d, and suppose that $a, b \in \mathbb{R}^n$ are such that $h(a)h(b) \neq 0$. The following are equivalent*

(i) *h is hyperbolic with respect to a, and $b \in \Lambda_{++}(a)$.*
(ii) *For all $x \in \mathbb{R}^n$, the polynomial*

$$(s,t) \mapsto h(x + sa + tb) \tag{1}$$

is stable.

Proof. Assume (ii). Let $x \in \mathbb{R}^n$. By (3) in Section 2 we see that all zeros of $s \mapsto h(x+sa)$ are real. Hence h is hyperbolic with respect to a. By setting $x = 0$ in (1) we see that $p(s,t) = h(sa+tb)$ is a homogeneous and stable polynomial of degree d in each variable. By (2) in Section 2 it follows that all zeros of $s \mapsto h(b + sa)$ are negative. Hence $b \in \Lambda_{++}(a)$.

Assume (i). Fix $s_0 \in \mathbb{H}$ and $x \in \mathbb{R}^n$ and consider the zero set, $Z(x)$, of $t \mapsto h(x + s_0 a + tb)$. We need to prove that $Z(x) \subset -\overline{\mathbb{H}} = \{z \in \mathbb{C} : \mathrm{Im}(z) \geq 0\}$ for all $x \in \mathbb{R}^n$. Consider $Z(0)$. Since $b \in \Lambda_{++}(a)$ all the zeros of $h(b + sa)$ are negative. Hence if $h(s_0 a + tb) = t^d h(b + s_0 t^{-1} a) = 0$, then $s_0/t < 0$. It follows that $Z(0) \subset -\overline{\mathbb{H}}$. To arrive at a contradiction assume that there is a vector $x \in \mathbb{R}^n$ for which $Z(x) \not\subset -\overline{\mathbb{H}}$. By moving from 0 to x along the line segment $\{\theta x : 0 \leq \theta \leq 1\}$, we see that for some $0 \leq \theta \leq 1$ we have $Z(\theta x) \cap \mathbb{R} \neq 0$ (by Hurwitz' theorem on the continuity of zeros). Hence there is a number $\alpha \in \mathbb{R}$ for which $h(\theta x + \alpha b + s_0 a) = 0$. Since $s_0 \notin \mathbb{R}$ and $\theta x + \alpha b \in \mathbb{R}^n$ this contradicts the hyberbolicity of h. □

Theorem 3. *Suppose that h is hyperbolic with respect to e.*

(i) *If $a \in \Lambda_{++}(e)$, then h is hyperbolic with respect to a, and $\Lambda_{++}(a) = \Lambda_{++}(e)$.*
(ii) *$\Lambda_{++}(e)$ is a convex cone.*

Proof. That h is hyperbolic with respect to a follows immediately from Lemma 2 since condition (ii) in Lemma 2 is symmetric in a and e. Since $a \in \Lambda_{++}(e)$, Proposition 1 implies $\Lambda_{++}(a) = \Lambda_{++}(e)$.

If $a, b \in \Lambda_{++}(e)$, then since $\Lambda_{++}(e) = \Lambda_{++}(a)$ it follows as in the last few lines of the proof of Proposition 1 that $ta + (1-t)b \in \Lambda_{++}(e)$, for all $0 \leq t \leq 1$. Clearly $\Lambda_{++}(e)$ is closed under multiplication of positive scalars. □

In order to understand hyperbolic programming we need to understand the nature of hyperbolicity cones. Suppose that h is hyperbolic with respect to e and of degree d. Then we may write

$$h(x + et) = \sum_{k=0}^{d} e_k(x) t^{d-k},$$

where $e_k(x)$ is a homogeneous polynomial of degree k. Now, $x \in \Lambda_+(e)$ if and only if all zeros of $t \mapsto h(x + et)$ are negative. Hence

$$\Lambda_{++}(e) = \{x \in \mathbb{R}^n : e_0(x) > 0, \dots, e_d(x) > 0\}.$$

Thus $\Lambda_{++}(e)$ is a semialgebraic set. Let Λ_+ be the closure of the hyperbolicity cone.

Proposition 4. *All faces of Λ_+ are exposed, that is, the faces are intersections with supporting hyperplanes.*

See [7] for a self-contained proof Proposition 4.

4. Are all hyperbolicity cones slices of the cone of PSD matrices?

Suppose that $A_1, \ldots, A_n$ are symmetric $d \times d$ matrices and $e = (e_1, \ldots, e_n)^T \in \mathbb{R}^n$. Suppose further that $\sum_{i=1}^n e_i A_i = I$, where I is the identity matrix. Then the polynomial

$$h(x) = \det(x_1 A_1 + \cdots + x_n A_n)$$

is hyperbolic with respect to e, and the hyperbolicity cone is

$$\Lambda_{++} = \{x \in \mathbb{R}^n : \sum_{i=1}^n x_i A_i \text{ is positive definite}\}.$$

Hence Λ_{++} is a slice of the cone of positive definite matrices with a hyperplane. Is this always the case?

Conjecture 5 (Generalized Lax conjecture)**.** *Suppose that $\Lambda_{++} \subseteq \mathbb{R}^n$ is a hyperbolic polynomial. Are there symmetric $d \times d$ matrices $A_1, \ldots, A_n$ such that*

$$\Lambda_{++} = \{x \in \mathbb{R}^n : \sum_{i=1}^n x_i A_i \textit{ is positive definite}\}.$$

The conjecture is true for $n = 3$ as demonstrated in [5], where it was shown to follow from the work of Helton and Vinnikov [4]. In fact in three variables more is true:

Theorem 6 ([4, 5])**.** *Suppose that $h(x, y, z)$ is of degree d and hyperbolic with respect to $e = (e_1, e_2, e_3)^T$. Suppose further that h is normalized such that $h(e) = 1$. Then there are symmetric $d \times d$ matrices A, B, C such that $e_1 A + e_2 B + e_3 C = I$ and*

$$h(x, y, z) = \det(xA + yB + zC).$$

The next lemma is due to Nuij [6].

Lemma 7. *The space of all degree d homogenous polynomials in n variables that are hyperbolic with respect to e has nonempty interior.*

A simple count of parameters shows that the exact analog of Theorem 6 does not hold in more than three variables. However some relaxations of the generalized Lax conjecture have been proposed. Two of these were disproved in [1].

5. Relaxations of hyperbolicity cones

Let $p \in \mathbb{R}[x_1, \ldots, x_n]$ and $e = (e_1, \ldots, e_n)^T \in \mathbb{R}^n$, and let

$$D[e](h) = \sum_{i=1}^n e_i \frac{\partial h}{\partial x_i}$$

denote the derivative in direction e.

Lemma 8. *Suppose that h is hyperbolic with respect to e. Then $D[e](h)$ is hyperbolic with respect to e, and its hyperbolicity cone contains the hyperbolicity cone of h.*

Proof. The proof just uses the following simple property of polynomials with only real zeros, which follows from Rolle's theorem: If $p(t)$ is a polynomial with only real zeros $\lambda_1 \leq \cdots \leq \lambda_n$, then the zeros $\lambda_1' \leq \cdots \leq \lambda_{n-1}'$ of $p'(t)$ are all real and satisfy

$$\lambda_1 \leq \lambda_1' \leq \cdots \leq \lambda_{n-1}' \leq \lambda_n. \tag{2}$$

Let $x \in \mathbb{R}^n$. We want to prove that the polynomial $q(t) = D[e](h)(x + et)$ has only real zeros. Let $p(t) = h(x + et)$. Clearly $q(t) = p'(t)$ so that $q(t)$ has only real zeros by the observation above. Also, by (2), we have $\lambda_1(x) \leq \lambda_1'(x)$, where $q(t) = C \prod_{i=1}^{n-1}(t + \lambda_i'(x))$. Hence

$$\{x : \lambda_1(x) > 0\} \subseteq \{x : \lambda_1'(x) > 0\},$$

as desired. □

Let h be a hyperbolic polynomial (with respect to e) of degree d. Define a sequence of hyperbolic polynomial $\{h_i\}_{i=0}^{d-1}$ by setting $h_0 = h$ and $h_{i+1} = D[e](h_i)$ for $0 \leq i \leq d-2$. Let Λ_{++}^i be the hyperbolicity cone of h_i. By Lemma 8

$$\Lambda_{++} = \Lambda_{++}^0 \subseteq \Lambda_{++}^1 \subseteq \cdots \subseteq \Lambda_{++}^{d-1},$$

and since h_{d-1} is homogeneous of degree 1, Λ_{++}^{d-1} is just a half-space.

6. The rank function of a hyperbolic polynomial

Clearly, the rank function on matrices satisfies $\mathrm{rank}(A) = \deg\det(I + tA)$. It is thus natural to define a rank function $\mathrm{rank}_h : \mathbb{R}^n \to \mathbb{N}$ associated to a hyperbolic polynomial with respect to e as follows:

$$\mathrm{rank}_h(x) = \deg h(e + xt).$$

Here are some properties of hyperbolic rank functions.

Proposition 9. *Let h be a hyperbolic polynomial with respect to e. Then*

(a) *The rank function does not depend on the choice of $e \in \Lambda_{++}$. ([1, 3, 7])*
(b) *The rank is constant on open line segments in Λ_+. ([7])*
(c) *The rank function is submodular on Λ_+, that is,*

$$\mathrm{rank}_p(u + v + w) + \mathrm{rank}_p(w) \leq \mathrm{rank}_p(u + w) + \mathrm{rank}_p(v + w),$$

for all $u, v, w \in \Lambda_+$. ([1, 3])

Proposition 9(c) enables us to define a class of polymatroids associated to a hyperbolic polynomial. A *polymatroid* on a finite set E is a function $r : 2^E \to \mathbb{N}$ such that

- $r(\emptyset) = 0$;
- If $S \subseteq T \subseteq E$, then $r(S) \leq r(T)$;
- r is *submodular*, that is,

$$r(S \cup T) + r(S \cap T) \leq r(S) + r(T),$$

for all subsets S and T of E.

Corollary 10. *Let h be a hyperbolic polynomial with respect to e and let $e_1, \ldots, e_n \in \Lambda_+$. Define a function $r : 2^{\{1,\ldots,n\}} \to \mathbb{N}$ by*

$$r(S) = \mathrm{rank}_h\left(\sum_{i \in S} e_i\right).$$

Then r is a polymatroid, called a hyperbolic matroid.

Let $V_1, \ldots, V_n$ be subspaces of a vectorspace V over a field K. Then the function $r : 2^{\{1,\ldots,n\}} \to \mathbb{N}$ defined by

$$r(S) = \dim\left(\sum_{j \in S} V_j\right)$$

is a polymatroid, where $\sum_{j \in S} V_j$ is the smallest subspace containing $\cup_{j \in S} V_j$. These are called K-*linear polymatroids*. It is not hard to see that the $\mathbb{R}$-linear matroids are exactly the hyperbolic matroids that come from the hyperbolic polynomial det. Are all hyperbolic matroids $\mathbb{R}$-linear? It turns out that the Vámos matroid V_8 is a hyperbolic matroid, but it is not K-linear for any K, see [1]. The reason for this is that the Vámos matroid fails to satisfy the so called Ingleton inequalities: Suppose that $r : 2^{\{1,\ldots,n\}} \to \mathbb{N}$ is a K-linear polymatroid. Then

$$\begin{aligned} &r(S_1 \cup S_2) + r(S_1 \cup S_3 \cup S_4) + r(S_3) + r(S_4) + r(S_2 \cup S_3 \cup S_4) \leq \\ &r(S_1 \cup S_3) + r(S_1 \cup S_4) + r(S_2 \cup S_3) + r(S_2 \cup S_4) + r(S_3 \cup S_4) \end{aligned}$$

for all $S_1, S_2, S_3, S_4 \in 2^{\{1,\ldots,n\}}$.

Apart from not satisfying the Ingleton inequalities, not much is known about the generality of hyperbolic polymatroids. It is desirable to get a better understanding of hyperbolic polymatroids, not only because of their role in the context of the generalized Lax conjecture.

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