A CRITERION FOR A GENERIC $m \times n \times n$ TO HAVE RANK n

Shuchao Bi

November, 2008

Abstract

Determining the rank of a tensor has always been an interesting and important problem in algebraic complexity theory[8], algebraic statistics[3, 9], engineering[4] and algebraic geometry[6]. In this paper, I will first give a criterion for a generic $m \times n \times n$ to have rank n. Right after the criterion, the first application is given . Then the symmetric version of this criterion is formulated. In Section 4, I will give a detailed discussion of the ($\mathcal{O}(1, 2)$ symmetric) ranks of ($\mathcal{O}(1, 2)$ symmetric) $3 \times 2 \times 2$ tensors over the complex and real numbers with the aid of these criterions. This criterion also provides another way to attack the "Salmon Problem" over real numbers.

1. The Criterion

In this section, we are working over any fixed field K. For a $m \times n \times n$ tensor X, let $X_1, X_2, \cdots, X_m$ (which are $n \times n$ matrices) denote the slices in the first direction.

Theorem. Let X be a $m \times n \times n$ tensor with X_1 nonsingular. Then X has rank n if the set of matrices

$$\{X_j X_1^{-1} : j = 2, ...m\}$$

can be diagonalized simultaneously.

Remark. The condition of the theorem can be weakened. In fact, if there exists a nonsingular linear combination of slices $X_1, X_2, \cdots, X_m$, then we can just replace X_1 by that linear combination. This operation doesn't change the rank at all. Note also that all linear combinations of X_i are singular is an algebraic condition, it amounts to say that $\det(\sum_{i=1}^{n} \lambda_i X_i) \equiv 0$ for all $\lambda_i \in \mathbb{C}$, i.e. all the coefficients of λ_i are zero. These are algebraic conditions on the entries of X.

Proof. Suppose the set of matrices
$$\{X_i X_1^{-1} : i = 2, ..., n\}$$
can be diagonalized simultaneously, that is, there exist invertible matrix P such that for $i = 2, \ldots , n$,
$$P X_i X_1^{-1} P^{-1} = E_i$$
where E_is are diagonal matrices.

Suppose
$$E_i = \begin{pmatrix} e_1^i & 0 & \cdots & 0 \\ 0 & e_2^i & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e_n^i \end{pmatrix}$$

For notation consistence, let $E_1 := I_n$. Then for $i = 1, ..., n$
$$X_i = P^{-1} E_i P X_1.$$

Suppose
$$P^{-1} = (a_1,\ a_2,\ \ldots\ ,\ a_n)$$
and
$$P X_1 = \begin{pmatrix} b_1^T \\ b_2^T \\ \vdots \\ b_n^T \end{pmatrix}$$
for a_i, $b_j \in K^n$.

Then
$$X_i = P^{-1} E_i P X_1 = (a_1,\ a_2,\ \ldots\ ,\ a_n) \begin{pmatrix} e_1^i & 0 & \cdots & 0 \\ 0 & e_2^i & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e_n^i \end{pmatrix} \begin{pmatrix} b_1^T \\ b_2^T \\ \vdots \\ b_n^T \end{pmatrix}$$

$$= e_1^i a_1 (b_1)^T + e_2^i a_2 (b_2)^T + ... + e_n^i a_n (b_n)^T.$$

The above formula implies X has rank $\leq n$, but X_1 is nonsingular implies X has rank $\geq n$. So we have X has rank exactly n. $\square$

2. First Application: the n-th secant variety of $\mathbb{P}^1 \times \mathbb{P}^{n-1} \times \mathbb{P}^{n-1}$ is not defective over $\mathbb{C}$

In this section, we work over complex numbers. Let $\mathbb{P}^1 \times \mathbb{P}^{n-1} \times \mathbb{P}^{n-1}$ be the Segre variety embedded in $\mathbb{P}^{2n^2-1}$. The expected dimension of the n-th secant variety $\sigma_n(\mathbb{P}^1 \times \mathbb{P}^{n-1} \times \mathbb{P}^{n-1})$ is

$$(n)(1+n-1+n-1)+n-1 = 2n^2-1.$$

In other words, we expect $\sigma_n(\mathbb{P}^1 \times \mathbb{P}^{n-1} \times \mathbb{P}^{n-1})$ to fulfill the ambient space $\mathbb{P}^{2n^2-1}$. In fact this is the case, as stated below.

Corollary. $\sigma_n(\mathbb{P}^1 \times \mathbb{P}^{n-1} \times \mathbb{P}^{n-1}) = \mathbb{P}^{2n^2-1}$.

Proof. Let

$$U_1 = \{X \in \mathbb{C}^2 \otimes \mathbb{C}^n \otimes \mathbb{C}^n : X_1 \text{ nonsingular and } X_2(X_1)^{-1} \text{ is diagonalizable}\}.$$

Let $\triangle$ denote the discriminant of $\det(X_2 - \lambda X_1)=0$, then the complement U_2 of $Z := \{$X: $\det X_1 = 0$, $\triangle=0$ $\}$ is contained in U_1 because $\triangle \neq 0$ implies $\det(X_2 - \lambda X_1)=0$ has n distinct roots in $\mathbb{C}$ and this implies $X_2(X_1)^{-1}$ is diagonalizable over $\mathbb{C}$. By the criterion given in the previous section, $U_1 \subseteq \sigma_n(\mathbb{P}^1 \times \mathbb{P}^{n-1} \times \mathbb{P}^{n-1})$, so $U_2 \subseteq \sigma_n(\mathbb{P}^1 \times \mathbb{P}^{n-1} \times \mathbb{P}^{n-1})$. But U_2 is a Zariski open set, take the Zariski closure of the above inclusion, we get $\mathbb{P}^{2n^2-1} \subseteq \sigma_n(\mathbb{P}^1 \times \mathbb{P}^{n-1} \times \mathbb{P}^{n-1})$ and finally they are equal. $\square$

3. The Symmetric Version

In this section, we work over real numbers and consider the Segre-Veronese embedding

$$\mathbb{P}(U) \times \mathbb{P}(V) \longrightarrow \mathbb{P}(U \otimes S^2 V)$$

with linear system $\mathcal{O}(1,2)$, where U, V are linear spaces of dimension m, n over $\mathbb{R}$. For a $\mathcal{O}(1,2)$ symmetric tensor $X \in \mathbb{P}(U \otimes S^2 V)$, let X_1, $X_2, \cdots, X_m$ (which are $n \times n$ symmetric matrices) denote the slices in the first direction. A decomposable tensor in $\mathbb{P}(U \otimes S^2 V)$ has the form $u \otimes v \otimes v$. Here X has rank n means k is the least number k such that

X can be written as the sum of k decomposable tensors in $\mathbb{P}(U \otimes S^2V)$.

Theorem. Let $X \in \mathbb{P}(U \otimes S^2V)$. Then rank$(X) \leq n$ if the set of matrices $\{X_i\}$ commute.

Proof. The X_is are symmetric, they commute implies they can be orthogonally diagonalized simultaneous, i.e. there exists an orthogonal matrix P such that $D_i := P^{-1}X_iP$ are diagonal for all i. Then $X_i = PD_iP^{-1} = PD_iP^T$. Suppose $D_i = \text{diag}\,\{d_{i1}, \ldots, d_{in}\}$, $P = (a_1, \ldots, a_n)$ where $a_i \in \mathbb{R}^n$, then

$$P^T = \begin{pmatrix} a_1' \\ \vdots \\ a_n' \end{pmatrix}$$

and

$$\begin{aligned} X_i &= (a_1, \ldots, a_n) \begin{pmatrix} d_{i1} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_{in} \end{pmatrix} \begin{pmatrix} a_1' \\ \vdots \\ a_n' \end{pmatrix} \\ &= d_{i1}a_1a_1' + \ldots + d_{in}a_na_n'. \end{aligned}$$

The above formula implies X has rank $\leq n$. $\square$

4. Ranks of $3 \times 2 \times 2$ Tensors

Let $R(l,n,m)_{\mathbb{R}}/R(l,n,m)_{\mathbb{C}}$ denote the maximal possible rank of $l \times m \times n$ tensors over real/complex numbers. Let $\underline{R}(l,n,m)$ denote the maximal broader rank (typical rank) of $l \times m \times n$ tensors over complex numbers.

For $2 \times 2 \times 2$ tensors, it is classically known that $R(2,2,2)_{\mathbb{R}} = R(2,2,2)_{\mathbb{C}} = 3$, see [7] for a simple proof. Over complex numbers, as stated in Section 3, $\underline{R}(2,2,2) = 2$. Over real numbers, somehow surprisingly, the two subsets of rank 2 and 3 tensors both have positive volume, as pointed out by Kusakal in [4]. A detailed analysis of ranks of $2 \times 2 \times 2$ tensors over $\mathbb{R}$ can be found in [1] and [4].

For $3 \times 2 \times 2$ tensors, it is also known that $R(3,2,2)_{\mathbb{R}} = R(3,2,2)_{\mathbb{C}} = 3$. Denote the slices of a nonzero $3 \times 2 \times 2$ tensor X in the first/second/third direction by X_1, X_2, X_3/Y_1, Y_2/Z_1, Z_2.

Proposition. For a nonzero $3 \times 2 \times 2$ tensor X, rank(X) $= 1$ if and only if
(1) All linear combinations of X_1, X_2, X_3 are singular, and
(2) Both Y_1 and Y_2 have rank ≤ 1 and
(3) Both Z_1 and Z_2 have rank ≤ 1.

Proof. "$\Longrightarrow$" If some linear combinations of X_1, X_2, X_3 are non-singular or some $\text{rank}(Y_j) = 2$ or some $\text{rank}(Z_k) = 2$, then trivially $\text{rank}(X) \geq 2$.

"$\Longleftarrow$" Suppose X is

$$\begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix}$$

with a, b, c, d, e, $f \in K^2$.

Up to permutation, we can assume $a \neq 0$. By conditions (1), (2), and (3) in this proposition, X has the following form:

$$\begin{pmatrix} a & \alpha_1 a & \alpha_2 a \\ \alpha_3 a & e & f \end{pmatrix}$$

with constants $\alpha_i \in K$. Then

$$X_1 = \begin{pmatrix} a \\ \alpha_3 a \end{pmatrix} \quad X_2 = \begin{pmatrix} \alpha_1 a \\ e \end{pmatrix} \quad X_3 = \begin{pmatrix} \alpha_2 a \\ f \end{pmatrix}$$

If $\alpha_1 \neq 0$, e must be a multiple of a because X_2 is singular. In the case $\alpha_1 = 0$, from the fact that $X_1 + X_2$ is singular, we also get that e is a multiple of a. So in any case, e is a multiple of a. Similarly, f is a multiple of a.

After possible row and column operations (which does not change the rank), we can further assume that $\alpha_1 = \alpha_2 = \alpha_3 = 0$, i.e. X becomes

$$\begin{pmatrix} a & 0 & 0 \\ 0 & \alpha_4 a & \alpha_5 a \end{pmatrix}$$

for some α_4, $\alpha_5 \in K$. Suppose

$$a = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

We know
$$\begin{pmatrix} a_1 & 0 \\ 0 & \alpha_4 a_1 \end{pmatrix}$$
is singular, so $\alpha_4 = 0$. Similarly, we get $\alpha_5 = 0$. That proves X has rank 1. □

to be continued...

References

1. Jos Berge, Kruskal's Polynomial for 2 x 2 x 2 Arrays and a Generalization to 2 x n x n Arrays, Psychometrika, v56 n4 p631-36, 1991.

2. T. W. Chaundy, On the number of real roos of a quintic equation, 1933.

3. Luis D. Garcia, Michael Stillman, and Bernd Sturmfels, Algebraic geometry of Bayesian networks, J. Symbolic Comput. 39 (2005), no. 3-4, 331C355. MR MR2168286 (2006g:68242)

4. J. B. Kruskal, Rank, decomposition, and uniqueness for 3-way and N-way arrays. In R. Coppi & S. Bolasco (Eds.), Multiway data analysis (pp. 7-18), 1989. Amsterdam: North-Holland.

5. Thomas Lickteig, Typical tensorial rank, Linear Algebra Appl. 69 (1985), 95C120. MR 87f:15017

6. G. Ottaviani, Symplectic bundles on the plane, secant varieties and Lüroth quartics revisited, preprint math.AG/0702151.

7. Toshio Sakata, Toshio Sumi, Mitsuhiro Miyazaki, A simple estimation of the maximal rank of tensors with two slices by row and column operations, symmetrization and induction, arXiv:0808.2688, 2008.

8. V. Strassen, Rank and optimal computation of generic tensors, Linear Algebra Appl. 52/53 (1983), 645C685. MR 85b:15039

9. B. Sturmfels, Open problems in algebraic statistics, arXiv:0707.4558, 2008.