Permanent vs. Determinant

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1 Introduction

A major problem in theoretical computer science is the Permanent vs. Determinant problem. It asks: given an n by n matrix of indeterminates $A = (a_{i,j})$ and an m by m matrix B with entries that are affine linear functions of the entries of A, what is the smallest m such that the determinant of B equals the permanent of A?

In other words, what is the complexity of writing the permanent in terms of the determinant? At first, one might believe that there should not be much difference because the two functions

$$\det A = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} a_{i,\sigma(i)}$$

and

$$\operatorname{perm} A = \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i,\sigma(i)}$$

appear to be very similar. However it is conjectured that $m > \operatorname{poly}(n)$ asymptotically. It is strongly believed that this conjecture is true because its falsity would imply $P = NP$. [1]

The best lower bound for the determinantal complexity of perm_n was obtained by studying the Hessian matrices of perm_n and $\det_m$. In this paper, we will let $T_p^{(2)}(x_{i,j})_{1\le i,j\le n}$ denote the n^2 by n^2 matrix with the entry in row i, j and column k, l equal to

$$\frac{\partial^2}{\partial x_{i,j} \partial x_{k,l}} \operatorname{perm}_n(x_{i,j})$$

and we will let $T_d^{(2)}(x_{i,j})_{1\le i,j\le n}$ denote the corresponding matrix for $\det_n$.

2 Lower Bound

Although the conjecture is that the permanent would have super polynomial determinantal complexity, the best lower bound attained so far is merely quadratic. This bound is due to Mignon and Ressayre. [2]

Theorem 2.1 *The determinantal complexity of perm_n is at least $n^2/2$.*

The proof of their result depends on the existence of an n by n matrix C with $\text{perm}_n(C)$ equal to 0 and $T_p^{(2)}(C)$ having full rank. We will assume that $n \geq 3$ since the case of $n = 2$ is trivial.

Lemma 2.2 *Let $n \geq 3$. Let C be an n by n matrix with $C_{1,1} = -n + 1$ and $C_{i,j} = 1$ for $(i, j) \neq (1, 1)$. Then $\text{perm}_n(C) = 0$ and $\text{rank}(T_p^{(2)}(C)) = n^2$.*

Proof By doing a Laplace expansion along the first row of C we get $\text{perm}_n(C) = -(n-1)P_{n-1} + (n-1)P_{n-1} = 0$, where P_k is the permanent of the k by k matrix with all entries equal to 1.

Note that P_k is equal to $k!$. We claim that

$$\frac{\partial^2}{\partial x_{i,j} \partial x_{k,l}} \text{perm}_n(C) = \begin{cases} (n-2)! & \text{if } 1 \in \{i, j, k, l\} \\ -2(n-3)! & \text{otherwise} \end{cases}$$

Indeed, differentiating perm_n with respect to $x_{i,j}$ and $x_{k,l}$ is equivalent to deleting rows i and k and columns j and l and taking the determinant of the remaining sub matrix. If $1 \in \{i, j, k, l\}$, then the remaining sub matrix is P_{n-2}. Otherwise, the remaining sub matrix is the $n-2$ by $n-2$ matrix with upper left entry equal to $1 - n$ and all other entries equal to 1.

Therefore, we can say

$$T_p^{(2)}(C) = (n-3)! \begin{bmatrix} 0 & B & B & \cdots & B \\ B & 0 & C & \cdots & C \\ B & C & 0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & C \\ B & C & \cdots & C & 0 \end{bmatrix}$$

where

$$B = (n-2) \begin{bmatrix} 0 & 1 & \cdots & 1 \\ 1 & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 1 & \cdots & 1 & 0 \end{bmatrix}$$

and

$$C = \begin{bmatrix} 0 & n-2 & n-2 & \cdots & n-2 \\ n-2 & 0 & -2 & \cdots & -2 \\ n-2 & -2 & 0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & -2 \\ n-2 & -2 & \cdots & -2 & 0 \end{bmatrix}$$

To show that $T_p^{(2)}(C)$ has full rank, we show that it has a trivial kernel. Since $B = (n-2)(J_n - I_n)$, then B only has eigenvalues $2-n$ and $(n-1)(n-2)$ so it has full rank.

Now let $Cv = 0$. By looking at the bottom two rows of C we have $(n-2)v_1 - 2v_2 - \cdots - 2v_{n-2} - 2v_n = 0$ and $(n-2)v_1 - 2v_2 - \cdots - 2v_{n-1} = 0$. This implies $v_{n-1} = v_n$. By a similar argument, $v_a = v_b$ for all $2 \leq a, b \leq n$.

By considering the first row of C, this implies that $v_2 = v_3 = \cdots = v_n = 0$. It follows that $v_1 = 0$ and $v = 0$. Therefore, both B and C have full rank.

Now let $\mathbf{x} = \begin{bmatrix} \vec{x_1} \\ \vec{x_2} \\ \vdots \\ \vec{x_n} \end{bmatrix}$ where each $\vec{x_i}$ is a vector in $\mathbb{R}^n$. Arguing similarly as we did for the matrix C we can show that $T_p^{(2)}(C)$ has full rank by first observing that $C(\vec{x_a} - \vec{x_b}) = 0$ for all $2 \leq a, b \leq n$. Since C has full rank, then $\vec{x_a} = \vec{x_b}$ for all $2 \leq a, b \leq n$. Thus $\vec{x_2} = \cdots \vec{x_n}$. By considering the first row of $T_p^{(2)}(C)$ we see that $\vec{x_a} = \vec{0}$ for all $2 \leq a \leq n$ because B has full rank. It follows that $\vec{x_1} = \vec{0}$ as well giving the desired result. □

We now present the proof of Mignon and Ressayre's lower bound using the matrix C from Lemma 2.2.

Proof (of Theorem 2.1) Let m be the determinantal complexity of perm_n. Then there exists a family of affine linear functions $A_{k,l}$, for $1 \leq k, l \leq m$ in the variables $x_{i,j}$ for $1 \leq i, j \leq n$, with $\text{perm}_n(x_{i,j}) = \det_m(A_{k,l}(x_{i,j})_{1 \leq i,j \leq n})$.

We can perform a translation on the coordinates $x_{i,j}$. By this we mean, there exist homogeneous linear functions $L_{k,l}$ and a matrix of constants Y such that $(A_{k,l}(x_{i,j}))_{k,l} = (L_{k,l}(x_{i,j} - C_{i,j})) + Y$. Thus

$$\text{perm}_n(x_{i,j}) = \det_m((L_{k,l}(x_{i,j} - C_{i,j})) + Y) \tag{2.1}$$

Wolog, we can apply a series of row and column operations to Y to put it in the form $\begin{bmatrix} 0 & 0 \\ 0 & I_s \end{bmatrix}$. Since $\text{perm}(C)$ is 0, then by equation (2.1), Y has determinant 0 and so does not have full rank. Thus $s < m$. Encode the row operations in a matrix P and the column operations in a matrix Q. Note that if we left multiply the above determinantal matrix by $\begin{bmatrix} \det P^{-1} & 0 \\ 0 & I_{m-1} \end{bmatrix} P$ and right multiply by $Q \begin{bmatrix} \det Q^{-1} & 0 \\ 0 & I_{m-1} \end{bmatrix}$ then we do not change the value of the determinant.

Since

$$\begin{bmatrix} \det P^{-1} & 0 \\ 0 & I_{m-1} \end{bmatrix} PYQ \begin{bmatrix} \det Q^{-1} & 0 \\ 0 & I_{m-1} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & I_s \end{bmatrix}$$

then wolog we can just let $Y = \begin{bmatrix} 0 & 0 \\ 0 & I_s \end{bmatrix}$ in equation (2.1).

By the multivariate chain rule, there exists an m^2 by n^2 matrix L such that

$$T_p^{(2)}(x_{i,j}) = L^T (T_d^{(2)}(L_{k,l}(x_{i,j} - C_{i,j}) + Y)_{k,l}) L$$

The matrix L has its (k, l, i, j) entry given by $\frac{\partial}{\partial x_{i,j}} L_{k,l}(x_{i,j} - C_{i,j})$.

Therefore

$$T_p^{(2)}(C) = L^T T_d^{(2)}(Y) L$$

Thus, $\text{rank}(T_p^{(2)}(C)) \leq \text{rank}(T_d^{(2)}(Y))$. By Lemma 2.2, $\text{rank}(T_p^{(2)}(C)) = n^2$. Therefore it suffices to show that $\text{rank}(T_d^{(2)}(Y)) \leq 2m$.

To see this, first consider the case when $s = m - 1$. Note that

$$\frac{\partial^2}{\partial x_{i,j} \partial x_{k,l}} \det_m Y$$

is non-zero if and only if one of the following holds:

1. $(i, j) = (1, 1)$ and $(k, l) = (t, t)$ for some $t > 1$
2. $(i, j) = (1, t)$ and $(k, l) = (t, 1)$ for some $t > 1$
3. $(i, j) = (t, 1)$ and $(k, l) = (1, t)$ for some $t > 1$
4. $(i, j) = (t, t)$ and $(k, l) = (1, 1)$ for some $t > 1$

The above four conditions tell us that $T_d^{(2)}(Y)$ has $3m - 2$ nonzero rows (1 row for condition 1 and $m - 1$ rows each for conditions 2, 3, and 4). Each of the $m - 1$ rows satisfying condition 4 are all copies of the same row with a 1 in column $(1, 1)$ and zeroes in every other column.

The $2m-2$ rows satisfying conditions 2 and 3 have only a single nonzero entry (of value 1) and each row contains a 1 in a different column. The row satisfying condition 1 contains $m-1$ nonzero entries. These nonzero entries are in columns that are not in the support of the rows satisfying conditions 2, 3, or 4. Thus, there are exactly $2m$ linearly independent rows in $T_d^{(2)}(Y)$.

Now consider the case when $s = m-2$. Note that

$$\frac{\partial^2}{\partial x_{i,j} \partial x_{k,l}} \det{}_m Y$$

is non-zero if and only if $i, j, k, l \in \{1, 2\}$. Then the number of non-zero entries in $T_d^{(2)}(Y)$ is at most 4 which is certainly less than $2m$ as n grows.

If $s < m-2$, then every entry of $T_d^{(2)}(Y)$ is 0 so we are done. $\square$

3 Upper Bounds

There is a combinatorial interpretation for perm_n. Let G be a digraph on n vertices labelled $\{1, 2, \ldots, n\}$ and let $(x_{i,j})_{1 \leq i,j \leq n}$ be the weighted adjacency matrix for G. Then $\mathrm{perm}(x_{i,j})$ is equal to the number of weighted directed cycle covers of G.

This is because

$$\mathrm{perm}(x_{i,j}) = \sum_{\sigma \in S_n} \prod_{i=1}^{n} x_{i,\sigma(i)}$$

and each cycle cover of G can be encoded as a permutation $\sigma \in S_n$ with $\sigma(i)$ identifying the directed edge $(i, \sigma(i))$ in G. The determinant can be interpreted similarly, but each cycle cover would be weighted by the sign of the permutation that it corresponds to. Thus, if every cycle cover of a graph corresponded to a permutation with even sign, then the permanent and the determinant would share this combinatorial interpretation.

Using this idea, Grenet was able to provide an upper bound for the determinantal complexity of the permanent. [3]

Theorem 3.1 *There exists a $2^n - 1$ by $2^n - 1$ matrix M with all entries of the form -1, 0, 1, or $x_{i,j}$ such that* $\det M = \mathrm{perm}_n(x_{i,j})$.

Proof Let G be a graph where the vertices are in bijection with subsets of $\{1, 2, \ldots, n\}$ but the vertices corresponding to $\emptyset$ and the whole set $\{1, 2, \ldots, n\}$ are identified as the same vertex v_0. It follows that G has $2^n - 1$ vertices.

Suppose vertices v, w correspond to $S, T \subseteq \{1, 2, \ldots, n\}$ respectively. We will put a directed edge with weight $x_{i,j}$ (with $1 \leq i, j \leq n$) between vertices v and w if and only if $|S| = i+1$, $|T| = i+2$, and $T \setminus S = \{j\}$. The vertex v_0 will have outgoing edges labelled $x_{n,j}$ and incoming edges labelled $x_{i,n}$. Now put loops with weight 1 on every vertex except v_0 to complete the edge set of G.

Note that every non-loop cycle in G has the form $x_{1,\sigma(1)}, x_{2,\sigma(2)}, \ldots, x_{n,\sigma(n)}$ since each cycle can be seen as the number of ways to add elements to the empty set until you have the set $\{1, 2, \ldots, n\}$. Since every vertex except v_0 contains a loop, then all cycle covers of G consist of non-loop cycles containing v_0 and loops. Thus, every cycle cover of G corresponds to an n-cycle. Note that the sign of an n-cycle is $(-1)^{n-1}$.

Let M be the adjacency matrix of G . It follows that the determinant of M equals

$$(-1)^{n-1} \sum_{\sigma \in S_n} x_{1,\sigma(1)} x_{2,\sigma(2)} \cdots x_{n,\sigma(n)}$$

which is equal to $\pm \mathrm{perm}_n(x_{i,j})$. If $\det M = -\mathrm{perm}(x_{i,j})$, then multiply the first row of M by -1 to get $\det M = \mathrm{perm}(x_{i,j})$. $\square$

Figure 1 shows the graph obtained from the proof of Theorem 3.1 in the case when $n = 3$. It is understood that the nodes labelled $\emptyset$ and $[3] = \{1, 2, 3\}$ are identified together and $\overline{S}$ refers to the complement of S in [3].

4 Glynn's Formula

We can consider alternative methods of rewriting the permanent besides expressing it as a determinant. Ryser was able to use inclusion-exclusion to write perm_n as a sum of $2^n - 1$ terms rather than a sum of $n!$ terms. Glynn was able to provide another exponential expression for perm_n, but he was able to write the polynomial as a sum of 2^{n-1} terms. [4]

Theorem 4.1 *We can write*

$$2^{n-1} perm_n = \sum_{\delta} \left(\prod_{k=1}^{n} \delta_i \right) \prod_{j=1}^{n} \sum_{i=1}^{n} \delta_i x_{i,j}$$

where the outer sum is over all 2^{n-1} vectors $\delta \in \{-1, 1\}^n$ with $\delta_1 = 1$.

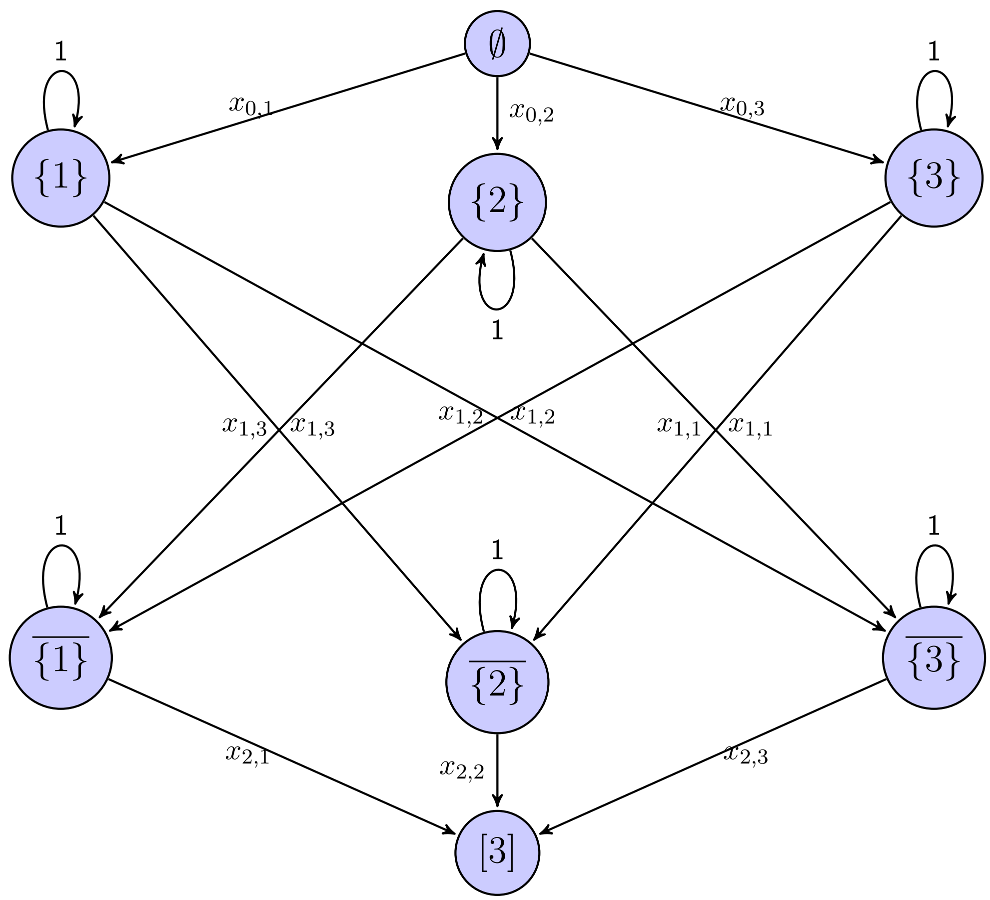

Figure 1: $n = 3$

Proof Consider a monomial m in the $x_{i,j}$ variables on the RHS of our proposed equality. Let λ_i be the degree of m in the variables $x_{i,j}$ for fixed i and varying j. Let the coefficient of m be c.

We have

$$c = \sum_{\delta} \prod_{i=1}^{n} \delta_i^{\lambda_i+1} = \prod_{i=2}^{n} \sum_{\delta_i \in \{-1,1\}} (\delta_i)^{\lambda_i+1}$$

Thus $c = 0$ unless λ_i is odd for all i. Note that m has total degree n. Thus $n = \sum_{i=1}^{n} \lambda_i$. Since $\lambda_1 = 1$, then $n - 1 = \sum_{i=2}^{n} \lambda_i$ and if λ_i is odd for all i, then we must have $\lambda_i = 1$ for all i.

Thus, the only such monomials m with non-zero coefficient would have coefficient 2^{n-1}. These monomials would have the form $\prod_{i=1}^{n} x_{i,\sigma(i)}$ where

$\sigma \in S_n$. It follows that the RHS of our proposed equality is equal to

$$\sum_{\sigma \in S_n} 2^{n-1} \prod_{i=1}^{n} x_{i,\sigma(i)}$$

which is exactly 2^{n-1}perm_n. $\square$

5 Fano Schemes

The Fano scheme of a space X, denoted $F_k(X)$, parameterizes k-dimensional planes that lie on X. Let D_n and P_n denote the space in P^{n^2-1} cut out by the n by n determinant and permanent respectively. Work by Chan and Ilten specifically studies the Fano schemes of the n by n determinant, $F_k(D_n)$, and the n by n permanent, $F_k(P_n)$. [5] It turns out the geometry of these schemes gives us information about the algebra of permanents.

Lemma 5.1 *It is impossible to write perm$_3$ as $l_1q_1 + l_2q_2$ where l_1, l_2 are linear forms in the variables $x_{i,j}$ and q_1, q_2 are quadratic forms in $x_{i,j}$.*

Proof Suppose that $\text{perm}_3 = l_1q_1 + l_2q_2$. Then the space Y cut out by l_1 and l_2 lies in the space cut out by perm$_3$. Since $Y \subseteq \mathbb{P}^8$ is cut out by two linear forms then it has codimension 2 which makes it a 6-dimensional space.

However, according to Table 2 in [5] the Fano scheme $F_k(P_3)$ is non-empty if and only if $k \leq 5$. The result follows by contradiction. $\square$

6 The case of $n = 3$

For this section we will let perm$_3$ be the permanent of $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$.

Grenet's work gives a 7 by 7 matrix whose determinant equals the 3 by 3 permanent.

$$\operatorname{perm}\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \det\begin{bmatrix} 0 & a & d & g & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & i & f & 0 \\ 0 & 0 & 1 & 0 & 0 & c & i \\ 0 & 0 & 0 & 1 & c & 0 & f \\ e & 0 & 0 & 0 & 1 & 0 & 0 \\ h & 0 & 0 & 0 & 0 & 1 & 0 \\ b & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

In conjunction with Theorem 2.1 and Theorem 3.1, we see that the determinantal complexity of perm_3 is 5, 6, or 7. It remains an open question to determine exactly which of the three numbers is the true determinantal complexity.

Another way of expressing of the permanent vs. determinant problem is by saying that we want to find matrices $C, A_{1,1}, A_{1,2}, \ldots, A_{3,3} \in \mathrm{M}(m)$ such that $M = C + x_{1,1}A_{1,1} + x_{1,2}A_{1,2} + \cdots + x_{3,3}A_{3,3}$ and

$$\operatorname{perm}_n = \det(M)$$

with m minimal. We can state a technical result that puts restrictions on the determinantal representation of perm_3.

Lemma 6.1 *Let $n = 3$. If $m = 6$, then the rank of C is 5. If $m = 5$, then the rank of C is 4.*

Proof We will prove only the case when $m = 6$. The case when $m = 5$ is similar. Note that by applying certain elementary row and column operations to M we do not change the value of its determinant. Row operations can be encoded as left multiplication by a matrix and column operations can be encoded as right multiplication by a matrix.

Suppose C has rank less than 3. Note that we can left multiply M by P and right multiply M by Q such that $PCQ = \mathrm{Diag}(1, 1, 0, 0, 0, 0)$. Then $\det M = \det PMQ$. Since PMQ is a matrix with two entries of the form $1 + l$ (where l is a linear form in the variables $(x_{i,j})$) and every other entry is a linear form in $(x_{i,j})$, then every term in the polynomial $\det PMQ$ has degree at least 4. Thus we cannot have $\operatorname{perm}_3 = \det M$.

If C has rank exactly 3, then let P and Q be such that $PCQ = \mathrm{Diag}(1, 1, 1, 0, 0, 0)$. It follows that the degree 3 part of $\det M = \det PMQ$ is equal to $\det M'$ where M' is the lower-right 3x3 sub matrix of PMQ. This implies that

perm_3 can be written as the determinant of a 3 by 3 matrix which is false because $\text{dc}(3) \geq 5$.

Now suppose that C has rank equal to 4. Let P and Q be such that $PCQ = \text{Diag}(1,1,1,1,0,0)$. The degree 2 part of $\det PMQ$ is equal to $\det M'$ where M' is the lower 2x2 sub matrix of PMQ. Thus $\det M' = 0$. We can apply further row operations to M' to transform it into the form $\begin{bmatrix} \alpha & \beta \\ 0 & 0 \end{bmatrix}$ where α and β are linear forms in the variables a,b,c,d,e,f,g,h,i. We can extend these row operations to PMQ so that wolog

$$PMQ = \begin{bmatrix} 1+d_1 & c_1 & c_2 & c_3 & e_1 & f_1 \\ c_4 & 1+d_2 & c_5 & c_6 & e_2 & f_2 \\ c_7 & c_8 & 1+d_3 & c_9 & e_3 & f_3 \\ c_{10} & c_{11} & c_{12} & 1+d_4 & e_4 & f_4 \\ g_1 & g_2 & g_3 & g_4 & \alpha & \beta \\ h_1 & h_2 & h_3 & h_4 & 0 & 0 \end{bmatrix}$$

where the indexed variables are linear forms in a,b,c,d,e,f,g,h,i.

The degree 3 part of $\det PMQ$ must be equal to perm_3. This gives us

$$\text{perm}_3 = \alpha(f_1h_1 + f_2h_2 + f_3h_3 + f_4h_4) + \beta(e_1h_1 + e_2h_2 + e_3h_3 + e_4h_4)$$

which contradicts Lemma 5.1.

Finally suppose that C has full rank. Let P and Q be such that $PCQ = I$. Then $\det M = \det PMQ$ would contain a constant term 1 so we could not have $\det M = \text{perm}_3$. The result follows. □

Glynn's formula (Theorem 4.1) allows us to write perm_3 as

$$(a+d+g)(b+e+h)(c+f+i) - (a-d+g)(b-e+h)(c-f+i) - \\ (a+d-g)(b+e-h)(c+f-i) + (a-d-g)(b-e-h)(c-f-i)$$

Another approach to determining the determinantal complexity of perm_3 might involve studying Glynn's formula, since he is able to express the permanent as a sum of 4 products of 3 linear terms rather than a sum of 6 products of 3 linear terms.

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