Math 239/Stat 260: Algebraic Statistics

Homework # 3
Due: Tuesday, September 30, 2008

1. Consider the following homogeneous rational function of degree zero:

$$L \quad = \quad \frac{p_{11}^2 \cdot p_{12}^3 \cdot p_{13}^5 \cdot p_{21}^7 \cdot p_{22}^{11} \cdot p_{23}^{13} \cdot p_{31}^{17} \cdot p_{32}^{19} \cdot p_{33}^{23}}{(p_{11}+p_{12}+p_{13}+p_{21}+p_{22}+p_{23}+p_{31}+p_{32}+p_{33})^{100}}$$

   - Maximize L over all (non-zero) non-negative 3×3-matrices (p_{ij}).
   - Maximize L over all non-negative 3×3-matrices (p_{ij}) of rank ≤ 1.
   - Maximize L over all non-negative 3×3-matrices (p_{ij}) of rank ≤ 2.

2. Consider the log-linear model described in Example 3.7 (and depicted in Figure 3) of the paper `arXiv:0803.1582` by E. Carlini and F. Rappalo.

   - Determine the dimension (degrees of freedom) of this model.
   - Compute the minimal Markov basis of this model.
   - Give a parametric representation of this model.
   - Determine the maximum likelihood degree of this model.

3. Show that every log-linear model $\mathcal{M}_A$ is an exponential family (as in Definition 2.3.11). For discrete data, does the coverse statement hold?

4. True or False: The Graver basis for the model of complete independence of m binary random variables consists of binomials of degree at most m.

5. The model in Example 2.2.10 is a hypersurface in the 7-dimensional simplex Δ_7. The defining polynomial is the $2\times2\times2$-hyperdeterminant. Compute the singular locus of this hypersurface, discuss its geometry (dimension, degree, irreducible?), and describe its statistical meaning.

6. Let G be the edge graph of an octahedron, so G is a graph with $m = 6$ vertices and 12 edges. Fix the corresponding Gaussian graphical model.

- Determine the maximum likelihood degree of this model.
- Suppose we are given data with sample covariance matrix

$$S \quad = \quad \begin{pmatrix} 55 & 40 & 31 & 28 & 31 & 40 \\ 40 & 55 & 40 & 31 & 28 & 31 \\ 31 & 40 & 55 & 40 & 31 & 28 \\ 28 & 31 & 40 & 55 & 40 & 31 \\ 31 & 28 & 31 & 40 & 55 & 40 \\ 40 & 31 & 28 & 31 & 40 & 55 \end{pmatrix}$$

  Compute the maximum likehood estimate $\hat{\Sigma}$ for these data.
- Can you express the entries of this matrix $\hat{\Sigma}$ in terms of radicals?