Math 113, Solutions to Midterm Exam # 2

(1) There is only one group of order 65, namely the cyclic group \( \mathbb{Z}/65\mathbb{Z} \). The proof uses the same argument as in Example 2.10.20 and Exercise 2.53, which goes as follows: Let \( G \) be any group of order 65 = 5 \( \cdot \) 13. The Third Sylow Theorem tells us that \( |\text{Syl}_5(G)| \) is in \( \{1, 13\} \) and is congruent to 1 modulo 5, and that \( |\text{Syl}_{13}(G)| \) is in \( \{1, 5\} \) and is congruent to 1 modulo 13. These facts imply that \( G \) has a unique Sylow 5-subgroup \( H_1 \) and a unique Sylow 13-subgroup \( H_2 \). Both \( H_1 \) and \( H_2 \) are normal by the Second Sylow Theorem. This implies that the map \( H_1 \times H_2 \to G, (h_1, h_2) \mapsto h_1h_2 \) is a group isomorphism. Since the \( H_1 \) and \( H_2 \) have prime order, they must be cyclic, and, using the Chinese Remainder Theorem, we conclude \( G \cong \mathbb{Z}/5\mathbb{Z} \times \mathbb{Z}/13\mathbb{Z} \cong \mathbb{Z}/65\mathbb{Z} \).

(2) Let \( a = (123) \) and \( b = (234) \). Then \( a, a^2, b \) and \( b^2 \) are four distinct three-cycles. Note that the three disjoint transpositions in \( A_4 \) can be written as follows:

\[
(12)(34) = ab, \quad (13)(24) = ba, \quad (14)(23) = ab^2a.
\]

Hence the subgroup of \( A_4 \) generated by \( a \) and \( b \) contains at least eight distinct elements, namely, \( a, a^2, b, b^2, ab, ba, ab^2a \) and the identity. Using Lagrange’s Theorem, we conclude that this subgroup must have order 12 and hence be equal to \( A_4 \).

(3) The greatest common divisor of 12 and 20 equals 4, and hence \( I = \langle 4 \rangle \). The greatest common divisor of 18 and 30 equals 6, and hence \( J = \langle 6 \rangle \). The sum ideal \( I + J \) is generated by gcd(4, 6) = 2, the intersection ideal \( I \cap J \) is generated by lcm(4, 6) = 12, and the product ideal \( I \cdot J \) is generated by 4 \( \cdot \) 6 = 24.

(4) (a) In the integers \( R = \mathbb{Z} \), the ideal \( I = \langle 3 \rangle \) is principal and maximal.

(b) In the integers \( R = \mathbb{Z} \), the ideal \( I = \langle 6 \rangle \) is principal and not prime.

(c) This is impossible because every maximal ideal is prime (Remark 3.2.8).

(d) In the polynomial ring over the integers, \( R = \mathbb{Z}[x] \), the ideal \( I = \langle 3, x \rangle \) is maximal but not principal. Reason for maximal: \( R/I \cong \mathbb{F}_3 \) is a field.

(e) In the polynomial ring over the integers, \( R = \mathbb{Z}[x] \), the ideal \( I = \langle x \rangle \) is principal and prime but not maximal. Reason: \( R/I \cong \mathbb{Z} \) is a domain but not a field.

(5) We compute \( x^4 + x^3 + 7 = (x^2 + x + 2)(x^2 - 2) + (2x + 11) \) and \( x^2 - 2 = (1/2x - 11/4)(2x + 11) + 113/4 \). Hence the answer in (a) is 2\( x + 11 \), and the answer in (b) is 113/4. These computations show that the unit 113/4 is in the ideal generated by \( f \) and \( g \), and hence \( \langle f, g \rangle = \mathbb{Q}[x] \).