Please start by writing your name and your student ID on the cover of your blue book. This exam is closed book. Do not use any notes, calculators, cell phones etc. Show all your work, and write full sentences if time permits. Each problem is worth 20 points, for a total of 100 points.

(1) Let $a, b$ be two integers and let $p$ be a prime number. Prove that

$$(a + b)^p \equiv a^p + b^p \pmod{p}.$$  

(2) Determine the set of all integers $x$ which satisfy the three congruences

$$x \equiv 1 \pmod{6}, \quad x \equiv 3 \pmod{10} \quad \text{and} \quad x \equiv 7 \pmod{15}.$$  

(3) Prove that every subgroup of an abelian group is normal.

(4) Consider the product of cyclic groups $G = \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z}$.

(a) Show that the group $G$ is cyclic.

(b) How many elements in $G$ are generators of $G$?

(c) How many elements in $G$ have order 10?

(5) The cycles $\sigma = (123)$ and $\tau = (124)$ are in the symmetric group $S_4$.

(a) Compute the two products $\sigma \tau$ and $\tau \sigma$ in $S_4$.

(b) Write both $\sigma \tau$ and $\tau \sigma$ as products of disjoint cycles.