Resolution of Singularity

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Abstract

We will follow the first two sections of Kirwan Chapter 7 to give us a description of the resolution of singularity $\tilde{C}$ of a sinuglar curve C. The first section defines the resolution of singularity and show that the nonsingular part of C and $\tilde{C}$ is holomorphic bijective. In the second section, Kirwan uses Newton polygon to get a sense of how the curve C looks like near a singular point and gives us a description of the fibre of a singular point of C in $\tilde{C}$.

1 Resolution of Singularity

In this section, Kirwan constructs a Riemann Surface $\mathcal{M}$ associated to the singular curve $\mathbb{C}$ to prove the following theorem:

Theorem 1 *$\tilde{C}$ is a compact connected Riemann surface. The map $\pi : \tilde{C} \to C$ is continuous and surjective. If C is nonsingular then π is a holomorphic bijection, and in general $\pi^{-1}(Sing(C))$ is finite and*

$$\pi : \tilde{C} - \pi^{-1}(Sing(C)) \to C - Sing(C)$$

is a holomorphic bijection.

$\tilde{C}$ is our resolution of singularity for C. The theorem above basically says that the resolution of singularity that we are going to definite is exactly what we want. It will "zoom in" into the singular points by giving us finite fibre at the singular points and keeping the other points bascially inact.

In order to construct $\tilde{C}$, we need to construct the Riemann Surface $\mathcal{M}$. Since the projective curves are defined by homogeneous polynomials, we can use

$$P(x, y(x), 1) = 0$$

to associate the curve with Riemann surfaces. Since the constructions is extremely long, I will outline the skeleton of the construction, and all the details can be found in the book. We start with the following definition so we can construct $\mathcal{M}$.

Definition 2 *An ordered pair of meromorphics functions defined on an open neighborhood of 0 in $\mathbb{C}$, (f,g), is simply called a pair if f is not constant on any neighbourhood of 0 and the map*

$$t \mapsto (f(t), g(t))$$

is one-to-one near zero. A parameter change is a holomorphic funtion ρ, is a holomorphic function defined on an open neighbourhood of 0 such that $\rho(0) = 0$, $\rho'(0) \neq 0$. Using the inverse function theorem from complex analysis, we can define a equivalence relation on the set of pairs. We say

$$(f, g) \sim (\tilde{f}, \tilde{g})$$

when there is a parameter change ρ such that $f \circ \rho = \tilde{f}$ and $g \circ \rho = \tilde{g}$ in some neightbourhood of 0. The equivalence class of a pair (f, g) is called a meromorphic element, denoted by $< f, g >$.

The underlying set of $\mathcal{M}$ is the set of meromorphic elements. Now we will define the topoloy on $\mathcal{M}$. Choose a pair (f, g) and let $r > 0$ be small enough so that f and g are both defined and meromorphic on the disk $D(0, r)$ of centre 0 and radius r and the map

$$t \mapsto (f(t), g(t))$$

is one-to-one on $D(0, r)$. We see that $(f(t_0 + t), g(t_0 + t))$ is a pair from the above definition (as a function of t) if $t_0 \in D(0, r)$. Now we define

$$U(f, g, r) = \{< f(t_0 + t), g(t_0 + t) >: t_0 \in D(0, r)\} \subseteq \mathcal{M}.$$

The following lemma gives us a topology on $\mathcal{M}$:

Lemma 3 *A subset is of $\mathcal{M}$ is open if and only if it is a union of subsets of the form $U(f, g, r)$ defined above. This is a topology on $\mathcal{M}$.*

The proof of the above lemma can be found on page 187. We now define an atlas of holomorphic charts on $\mathcal{M}$

$$\{\phi_\alpha : U_\alpha \to V_\alpha : \alpha \in \mathcal{A}\}$$

to make $\mathcal{M}$ a Riemann surface.

Definition 4 *Let $\mathcal{A}$ be the set of all ordered triples (f,g,r) where (f,g) is a pair and $r > 0$ is small enough so that f and g are defined and meromorphic on $D(0, r)$ and the mapping*

$$t \mapsto (f(t), g(t))$$

is one-to-one on $D(0, r)$.

Definition 5 *If $\alpha = (f, g, r) \in \mathcal{A}$ let*

$$U_\alpha = U(f, g, r)$$

and

$$V_\alpha = D(0,r)$$

and let $\phi_\alpha : U_\alpha \to V_\alpha$ *be the inverse of the homeomorphism*

$$\theta_\alpha : t_0 \mapsto < f(t_0 + t), g < t_0 + +t >$$

θ_α is a homoemorphism by Lemma 7.4 (on page 188) in Kirwan's book.

Proposition 6 $\mathcal{M}$ *is a Riemann surface with the holomorphic atlas*

$$\Phi = \{\phi_\alpha : U_\alpha \to V_\alpha : \alpha \in \mathcal{A}\}$$

defined above.

The proof that the induced transition functions are holomorphic is given on page 189 and the proof that $\mathcal{M}$ is Hausdorff is given in page 201.

We can now relate $\mathcal{M}$ to our projective curves.

Definition 7 *Let P(x,y,z) be a nonconstant irreducible homogeneous polynomial of degree d not divisible by z. The Riemann surface* S_p *of P(x,y,z) is the open subset of* $\mathcal{M}$ *consisting of all those elements* $< f, g >$ *of* $\mathcal{M}$ *satisfying*

$$P(f(t), g(t), 1) = 0$$

for all t in some neighbourhood of 0. If C is the projective curve

$$C = [x, y, z] \in \mathbb{P}_2 : P(x, y, z) = 0$$

then we write $\tilde{C}$ *for* S_p *and define* $\pi : \tilde{C} \to C$ *by*

$$\pi(< f, g >) = [\tilde{f}(0), \tilde{g}(0), 0]$$

where $\tilde{f}(t) = t^n f(t)$ *and* $\tilde{g}(t) = t^n f(g)$ *and n is the multiplicity of the pole at 0 of f or g, whichever is greater.*

Using the definition above, we have finally constructed a resolution of singularity of the singular curve C and we can thus make sense of the theorem we give in the beginning of the paper. By just looking at the definition, it is not suprising that the Theorem holds. The part where f and g are both holomorphic gives us the bijection between the nonsingular parts of $\tilde{C}$ and C and the other part will in turn gives us the fibre on the singular points, as we can easily see that functions with different mulitiplicity at the pole can be sent to the function $\tilde{f}$ or $\tilde{g}$ as long as their holomorphic part is the same.

Note that our way to construct a resolution of singularity is not the only way, in fact, there are quite a few other methods. For example, the blow-up we have encounted in class is one of the ways to resolve singularity.

We will investigate the singular points using Newton's method next.

2 Newton polygons and Puiseux expansions

We will simplify our notation by projectively transforming the singular point to [0,0,1]. The main result from this section is the following:

There exists $m_1, ..., m_k \in \mathbb{Z}_+$ and power series in x^{1/m_j} (Puiseux expansions)

$$\sum_{r \geq 1} a_r{}^{(j)} x^{r/m_j}$$

for $i \leq j \leq k$ such that if x and y are near 0 then the Puiseux expansions converge and

$$P(x, y, 1) = 1$$

if and only if

$$y = \sum_{r \geq 1} a_r{}^{(j)} (x^{1/m_j})^r$$

for some $j \in \{1, ..., k\}$ and some choice of m_jth root x^{1/m_j} of x.

It basically says that if x, y so that $P(x, y, 1) = 0$, y must be a series in fractional powers of x. Newton used a method of calculating Puiseux expansions via polygons. Before defining Newton polygon, we need to define the carrier of a polynomial.

Definition 8 *Let*

$$P(x, y, 1) = \sum_{\alpha,\beta} c_{\alpha\beta} x^\alpha y^\beta,$$

then the carrier, $\triangle(P)$*, of* $P(x, y, 1)$ *is*

$$\triangle(P) = \{(\alpha, \beta) \in \mathbb{Z}^2 : c_{\alpha\beta} \neq 0\}$$

Definition 9 *If* $p, q \in \mathbb{R}^2$ *let*

$$[p, q] = \{tp + (1 - t)q : 0 \leq t \leq 1\}$$

be the line segment from p *to* q*. Newton polygon of* P *is the boundary of the convex subset of* $\mathbb{R}^2$ *consisting of those* $(x, y) \in \mathbb{R}^2$ *such that*

$$x \geq a \text{ and } y \geq b$$

for some $(a, b) \in [\delta_1, \delta_2]$ *where* δ_1 *and* δ_2 *belong to the carrier* $\triangle(P)$*.*

Notice from the definition the Newton polygon must consist of a vertical half-line and a horizontal half-line.

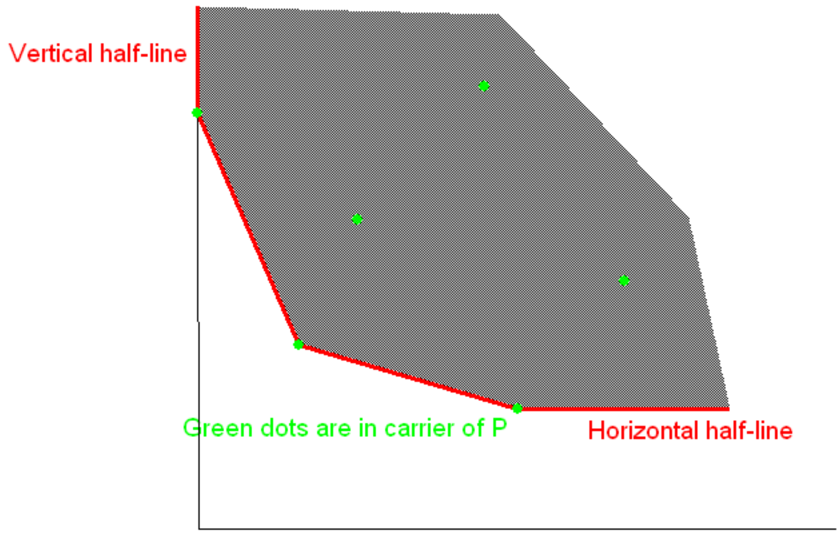

Figure: Newton Polygon

So what do we do with this Newton polygon? Notice that we can always choose coordinates such that $P(x, y, z)$ is not divisible by x, so we have

$$(0, \beta) \in \triangle(P).$$

The case where the Newton polygon is one point (it must be our point), we have a trivial solution $y = 0$. For the nontrivial cases, we start with the steepest segment and let $(0, \beta_0)$ be the upper endpoint of the segment (convexness ensures it has endpoint of this form) and let $-\frac{1}{\mu_0}$ be its slope. Since the points in the carrier of P must be integer valued, the slope is rational (also positive by the construction of Newton polygon). Say

$$\mu_0 = \frac{p_0}{q_0}$$

where p_0 and q_0 positive comprime integers. We then write

$$P(x, y, 1) = \sum_{\alpha_0 + \mu_0 \beta \geq \nu_0} c_{\alpha\beta} x^\alpha y^\beta$$

where

$$\nu_0 = \mu_0 \beta_0$$

We want to "pull out" the parts where $\nu_0 = \alpha + \mu_0 \beta$. Clearly, $(0, \beta_0)$ satisfies the above equation. We can also find another point satisfying the above, namely the other point our chosen line segment. The slope of our chosen line is

$$\frac{\beta_0 - \beta}{-\alpha} = -\frac{1}{\mu_0}.$$

Rearrange it and we get

$$\mu_0\beta_0 = \alpha + \mu_0\beta$$

which tells us precisely that the other point satisfies our condition. With these, we can "pull out" the part where $\nu_0 = \alpha + \mu_0\beta$ and write

$$P(x, tx^{\mu_0}, 1) = x^{\nu_0} f_0(t) + \sum_{\alpha+\mu_0\beta>\nu_0} c_{\alpha\beta} x^\alpha y^\beta$$

where

$$f_0(t) = \sum_{\alpha+\mu_0\beta=\nu_0} c_{\alpha\beta} t^\beta$$

Since f_0 only has one variable, it has a nonzero root, say t_0. Making

$$y_0 = t_0 x^{\mu_0}$$

gives us the first approximate solution to the equation

$$P(x, y, 1) = 0$$

by making the ν_0 term vanishes. To continue, we next make the substituion

$$x = (x_1)^{q_0} \text{ and } y = {x_1}^{p_0}(t_0 + y_1)$$

We get

$$P(x, y, 1) = {x_1}^{q_0\nu_0} P_1(x_1, y_1)$$

where

$$P_1(x_1, y_1) = \sum_{q_0\alpha+p_0\beta\geq q_0\nu_0} c_{\alpha\beta} {x_1}^{q_0\alpha+p_0\beta-q_0\nu_0}(t_0 + y_1)^\beta$$

We can just repeat the process we did for P(x,y,1) and continue to get approximate solutions (x_i, y_i)'s. We will thus get an expansion of y by summing up all the y_i's. The series we get from this process is call a Puiseux expansion for the curve

$$C = \{[x, y, z] \in \mathbb{P}_2 : P(x, y, z) = 0\}$$

The next theorem shows the properties of Puiseux expansion, as stated in the beginning of the section.

Theorem 10 *Any Puiseux expansion*

$$y = \sum_{r\geq 1} a_r x^{r/n}$$

for the curve C near the point [0,0,1] is a power series in $x^{1/n}$ which converges for x sufficiently close to 0 and satisfies

$$P(x, \sum_{r\geq 1} a_r x^{r/n}, 1) = 0$$

The above theroem thus conclues this section as it gives us exactly what we want, i.e., the Puiseux expansion near the singular point [0,0,1] give us a picture of what the points near the singularity look like by giving us a description of how y behaves given an x. Before we end the paper, we will look at the fibre of our Riemann surface $\tilde{C}$ at [0,0,1], our singular point. We have

$$\pi^{-1}\{[0,0,1]\} = \{< t^{m_j}, g_j(t) >: 1 \leq j \leq i, g_j(0) = 0\}$$

where the Pusieux expansion of C near [0.0,1] are given by

$$y = g_j(e^{2\pi is/m_j} x^{1/m_j}), 1 \leq j \leq l, 1 \leq s \leq m_j, g_j(0) = 0$$

We know that every element $< f, g >$ of $\mathcal{M}$ can be express in one of the forms:

$$< a + t^m, g(t) > \text{ if f holomorphic at 0 and } f(t) - c_0 \text{ has multiplicity m at 0}$$

and

$$< t^{-m}, g(t) > \text{ if f has poles of order m at 0}$$

from Remark 7.10 in page 190. This tells us that our $\pi^{-1}\{[0,0,1]\}$ does indeed has the form stated above because $a + t^m = 0$ when $t = 0 \implies a = 0$

Two Puiseux expansions are essentially different if the j's are different, i.e. we are taking different order roots. By the way we define the meromorphic elements in $\mathcal{M}$, we see that the points in the inverse image of [0,0,1] in $\tilde{C}$ are given by the essentially different Puisueux expansions near [0,0,1]. Thus computing the Puiseux expansions will tell us how the fibre of the singular points look like.