Score 5.5

Homework 7

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Problem 5.3

1

Let B be the original optimal basis matrix where

$$B = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1m} \\ b_{21} & b_{22} & \cdots & b_{2m} \\ \vdots & \vdots & & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mm} \end{pmatrix}$$

First, we are going to find $B(\delta)^{-1}$ since we need to check feasibility and optimality conditions for the matrix $B(\delta)$ for different values of δ. Note that

$$B(\delta) = \begin{pmatrix} b_{11}+\delta c_1 & b_{12} & \cdots & b_{1m} \\ b_{21}+\delta c_2 & b_{22} & \cdots & b_{2m} \\ \vdots & \vdots & & \vdots \\ b_{m1}+\delta c_m & b_{m2} & \cdots & b_{mm} \end{pmatrix}$$

where the vector $A_0' = (c_1, \ldots, c_m)'$. Let's denote B^{-1} as

$$B^{-1} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mm} \end{pmatrix}$$

which is the inverse matrix of B, and it exists because B is an optimal basis matrix. Since we are considering the matrix $B(\delta)$ only in the context where $\delta \in [\delta_1, \delta_2]$ in which the determinant of $B(\delta)$ is nonzero, we are going to say that the matrix $B(\delta)$ is invertible and most of the time we are not going to mention the context. Let $v' = (\delta\ 0\ \ldots\ 0)' \in \mathfrak{R}^m$. Note that

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \end{pmatrix} \times \begin{pmatrix} b_{11}+\delta c_1 & b_{12} & \cdots & b_{1m} \\ b_{21}+\delta c_2 & b_{22} & \cdots & b_{2m} \\ \vdots & \vdots & & \vdots \\ b_{m1}+\delta c_m & b_{m2} & \cdots & b_{mm} \end{pmatrix} \neq \mathbf{0}$$

because $(a_{11}\ a_{12}\ \dots\ a_{1m})' \neq \mathbf{0}$ since B^{-1} is invertible, and the rows of $B(\delta)$ are linearly independent since $B(\delta)$ is invertible. However,

$$(a_{11} \quad a_{12} \quad \cdots \quad a_{1m}) \times \begin{pmatrix} b_{11}+\delta c_1 & b_{12} & \cdots & b_{1m} \\ b_{21}+\delta c_2 & b_{22} & \cdots & b_{2m} \\ \vdots & \vdots & & \vdots \\ b_{m1}+\delta c_m & b_{m2} & \cdots & b_{mm} \end{pmatrix} = (1+v'B^{-1}A_0\ 0\ \dots\ 0)$$

This implies that $1+v'B^{-1}A_0 \neq 0$. Consequently, we can apply the Sherman-Morrison formula to find $B(\delta)^{-1}$. Thus,

$$B(\delta)^{-1} = (B+A_0v')^{-1} = B^{-1} - \frac{B^{-1}A_0v'B^{-1}}{1+v'B^{-1}A_0}$$

where A_0v' is the outer product of two vectors A_0 and v. Note that for $\delta = 0$, $B(\delta)^{-1} = B^{-1}$. Consequently, for $\delta = 0$, $B(\delta)$ is an optimal basis. Let γ be a real value such that $B(\gamma)$ is an optimal basis. Assume, without loss of generality, that $\gamma > 0$. Since $B(0)$ and $B(\gamma)$ are optimal basis, $B(0)^{-1}b \geq 0$ and $B(\gamma)^{-1}b \geq 0$ since they represent feasible solutions. Note that

$$B(\delta)^{-1}b = \begin{pmatrix} x_1 - \frac{x_1\delta H}{1+\delta H} \\ x_2 - \frac{x_1[\delta a_{21}c_1+\ldots+\delta a_{2m}c_m]}{1+\delta H} \\ \vdots \\ x_m - \frac{x_1[\delta a_{m1}c_1+\ldots+\delta a_{mm}c_m]}{1+\delta H} \end{pmatrix}$$

where $H = a_{11}c_1+\ldots+a_{1m}c_m$ and $(x_1 \ldots x_m)$ is the optimal solution associated with the basis matrix B. Note that the function $g_j(\delta) = x_j - \frac{x_1\delta[a_{j1}c_1+\ldots+a_{jm}cm]}{1+\delta H}$ is a continuous real valued function since $1+\delta H \neq 0$ for any δ, and we know that $g_j(0) \geq 0$ and $g_j(\gamma) \geq 0$. Let $0 \leq \alpha \leq \gamma$. Assume, by way of contradiction, that $g_j(\alpha) < 0$ for some $0 \leq j \leq m$. By Bolzano's theorem, there exist two points a_1 and a_2 between 0 and γ such that $g_j(a_1) = g_j(a_2) = 0$ which implies that there exists a point $a_1 < \phi < a_2$ such that $g_j'(\phi) = 0$ by Rolle's theorem. Thus,

$$0 = g_j'(\phi) = \frac{-[x_1(a_{j1}c_1+\ldots+a_{jm}c_m)(1+\phi H) - Hx_1\phi(a_{j1}c_1+\ldots+a_{jm})]}{denominator}$$

which implies that

$$x_1(a_{j1}c_1+\ldots+a_{jm}c_m)(1+\phi H) = Hx_1\phi(a_{j1}c_1+\ldots+a_{jm})$$

If $x_1 = 0$, then $B(\delta)^{-1}b = B^{-1}b$ and we're done. $B(\delta)^{-1}b$ is a continuous function of δ. Moreover, if $a_{j1}c_1+\ldots+a_{jm} = 0$, then $g_j(\phi) = 0$, and we're done. Then from this point forward we can assume that $x_1 \neq 0$ and $a_{j1}c_1+\ldots+a_{jm} \neq 0$ which implies that we can cancel these terms in the previous equation getting as a result

$$1+\phi H = H\phi$$

but this is impossible. So, our hypothesis is untenable which implies that $B(\delta)^{-1}b \geq 0$ for $0 \leq \delta \leq \gamma$. In order to prove optimality conditions for $B(\delta)^{-1}$, we are going to do the same. First, let's compute $d' - d'_{B(\delta)}B(\delta)^{-1}A$.

$$d' - d'_{B(\delta)}B(\delta)^{-1}A = d' - d'_{B(\delta)}(B^{-1} - \frac{B^{-1}A_0v'B^{-1}}{1 + v'B^{-1}A_0})A$$

where d is the cost vector. As you can check, $d' - d'_{B(\delta)}B(\delta)^{-1}A$ is

$$d' - d'_{B(\delta)} \begin{pmatrix} 1 - \frac{\delta\sum_{j=1}^{m} a_{1j}c_j}{a+\delta H} & 0 & \dots & 0 & d_{1(m+1)} - \frac{\delta d_{1(m+1)}\sum_{j=1}^{m} a_{1j}c_j}{1+\delta H} & \dots \\ -\frac{\delta\sum_{j=1}^{m} a_{2j}c_j}{a+\delta H} & 1 & \dots & 0 & d_{2(m+1)} - \frac{\delta d_{1(m+1)}\sum_{j=1}^{m} a_{2j}c_j}{1+\delta H} & \dots \\ \vdots & 0 & \vdots & 0 & \vdots & \dots \\ -\frac{\delta\sum_{j=1}^{m} a_{mj}c_j}{a+\delta H} & 0 & \vdots & 1 & d_{m(m+1)} - \frac{\delta d_{1(m+1)}\sum_{j=1}^{m} a_{mj}c_j}{1+\delta H} & \dots \end{pmatrix}$$

After we got this matrix, the idea is the same. Define a function g_j for each row in the matrix defined by $d' - d'_B B(\delta)^{-1}A$. These functions $g's$ are continuous real valued functions. So, if one of these functions is strictly less than 0 for some value α between 0 and γ is because this function g has two zeros, and between these two zeros there exists a zero of its derivative. After this, we just need to check that it is impossible to find a zero of the derivative of the function g between 0 and γ.

Therefore, the subset of $[\delta_1, \delta_2]$ for which $B(\delta)$ is an optimal basis is also a closed interval. ■

Problem 5.8

(a)
The optimal quantity of each service set is as follows; JJP English: 0, Currier: 2, Primrose method 1: 0, Primrose method 2: 0, and Bluetail: 5. The total profit is $102 * 2 + 89 * 5 = 649$.

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(b)
Increasing the pounds of clay available in 23.33 results in a revenue increase of $23.33 \times 1.429 = 33.33857$. Similarly, increasing the kiln time in 5.60 hours results in a revenue increase of 112.8008. However, increasing the pounds of enamel and the hours dry room have no effect on revenues (the associated dual variables are zero).

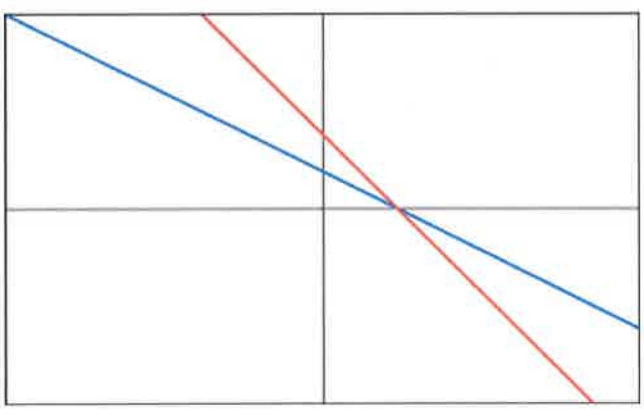
(c)
Yes. According to our previous description, increasing the pounds of clay in 20 lbs. results in a revenue increase of $20 \times 1.429 = 28.58$. Of course, we need to subtract the cost of the extra 20 lbs. of clay which is $20 \times 1.1 = 22$. The profit associated to the additional 20 lbs. of clay is $28.58 - 22 = 6.58$.

(d)
According to **Table 5.5**, we can decrease the number of hours available in the dry room by 28, and this does not have effect on the revenues. However, when the number of hours available in the dry room decreases by 30 (2 more than the maximal allowance), we need to consider the worst-case-scenario in order to give a bound for the decrease in the total profit. Note that according to **Table 5.3**, Currier room service set is the one that generates more profit and it is also the one that consumes less dry room hours. Consequently, our worst-case-scenario is precisely when we produce two less Currier room service sets which generates a lost of 204. Therefore, the bound for the decrease in the profit is 204.

(e)
Yes, actually we compute the solution including the new constraint, and the optimal solution is JJP English:0, Currier:1, Primrose method 1:3.5, Primrose method 2:0, and Bluetail:4. This happens because according to **Table 5.5** increasing the production Primrose in 3.50 results in a revenue increase of 40.0015.
■

Problem 5.10

(a)



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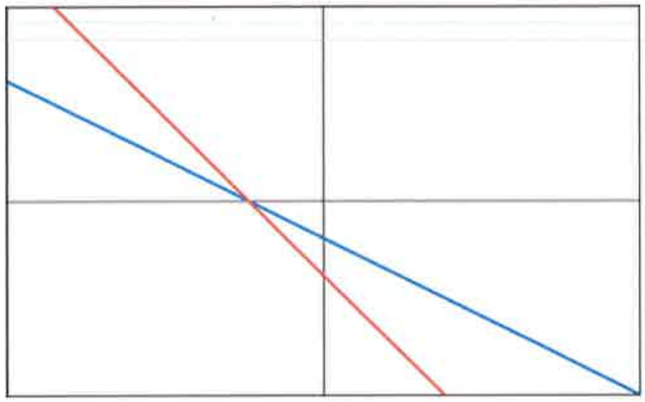
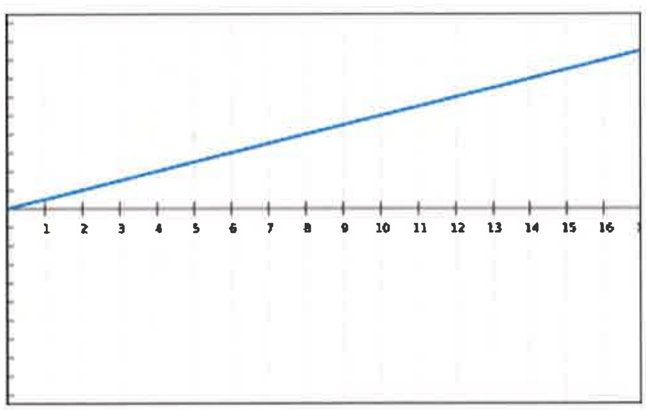
In the figure, the blue line represents the equation $x_1 + 2x_2 = \theta$, and the red line represents $x_1 + x_2 = \theta$ where $\theta > 0$. Note that the feasible set is precisely the portion of the blue line that lies in the first quadrant. Since we are minimizing, we need to move the line in the direction of the vector $(-1, -1)$ without leaving the feasible region. In this way, we find the optimal solution which is $x_1 = 0$ and $x_2 = \theta/2$. Consequently, the optimal value is $\theta/2$.



In the figure, the blue line represents the equation $x_1 + 2x_2 = \theta$, and the red line represents $x_1 + x_2 = \theta$ where $\theta < 0$. As you can see, the feasible set is empty which implies that the problem is infeasible. To conclude, note that for $\theta = 0$, the only feasible point is $(0, 0)$. Consequently, in this case the optimal value is 0.

(b)



The blue line represents the function $f(\theta) = \theta/2$. The function is defined for positive values of θ because for $\theta < 0$ the problem is infeasible.

(c)
In virtue of **Theorem 5.2**, the set of all dual optimal solutions for $\theta > 0$ is the set whose only element is $1/2$. Indeed, $1/2$ is the only subgradient of the optimal cost function f at any point $\theta > 0$. Similarly, for $\theta = 0$, the set of all dual optimal solutions is $\{p \in \Re \mid p \leq 1/2\}$ since these are all the vectors that are subgradient of the optimal cost function f at 0. However, for $\theta < 0$ we cannot apply **Theorem 5.2** since for $\theta < 0$ the primal problem is infeasible.

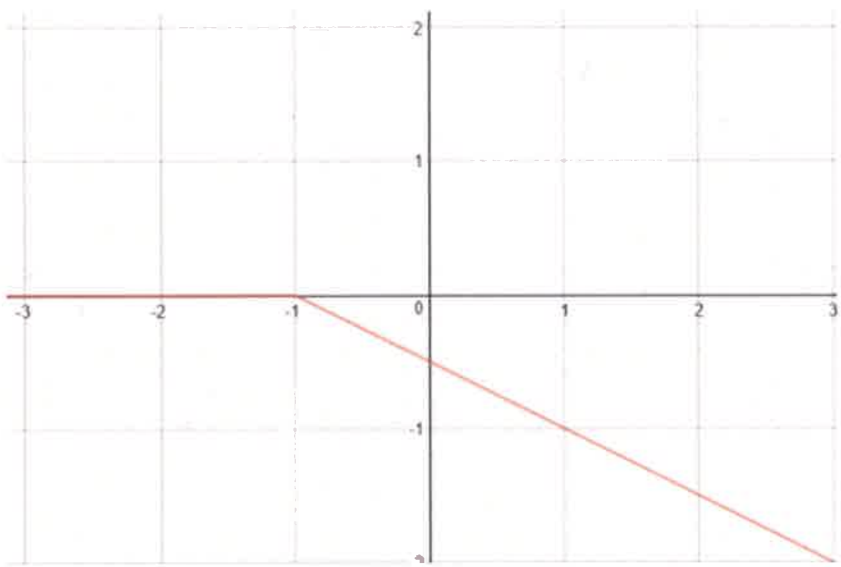
We know if the primal problem is infeasible, then the dual is either unbounded or infeasible. Either case the set of all dual optimal solutions is the empty set. ■

Problem 5.11

No, it is not true. Consider the following counterexample.

$$\begin{aligned} \text{minimize } & (c+\theta d)'x \\ \text{subject to } & x_1 + x_2 = 1 \\ & x_1, x_2 \geq 0. \end{aligned}$$

where $c' = d' = (-1/2 \; -1/2)$. Note that the extreme points of this linear optimization problem are $x^1 = (0,0)$, $x^2 = (1,0)$, and $x^3 = (0,1)$. Consider the function $g(\theta) = \min_{i=1,2,3}(c+\theta d)x^i = \min\{0, -1/2 - 1/2\theta, -1/2 - 1/2\theta\}$. We represent $g(\theta)$ in the following figure. 1



Note that $g(\theta)$ is linear for $\theta \in [0,2]$. However, it is not the case that there exists a unique optimal solution for $0 < \theta = 1 < 2$. Indeed, for $\theta = 1$ the problem has infinite optimal solutions, namely, all the points in the portion of the line $x_1 + x_2 = 1$ lying in the first quadrant. ■

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Problem 5.13

(a)
The initial tableau is

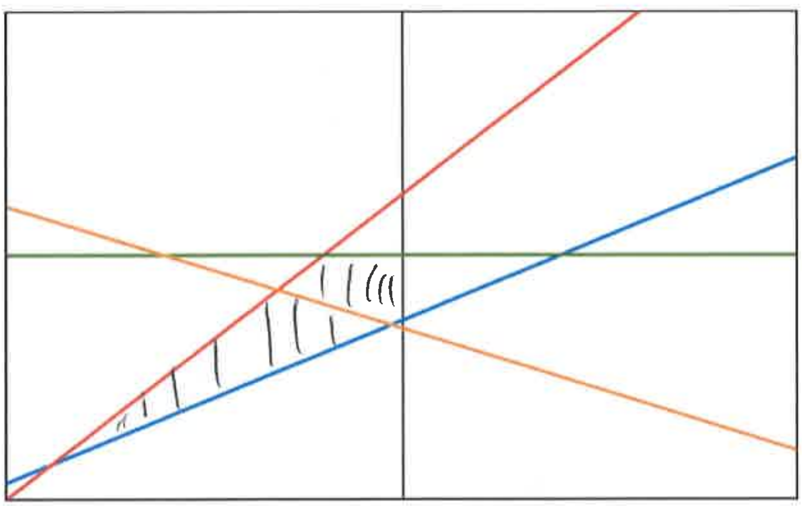
		x_1	x_2	x_3	x_4
	0	4	0	5	0
$x_2 =$	1	2	1	-5	0
$x_4 =$	2	-3	0	4	1

Note that $x_2 = 1$ and $x_4 = 2$ are valid initial basic variables because the second and the fourth column are linearly independent, and their values are greater than zero. Since the reduced costs are non-negative, our solution is optimal which implies that the optimal cost is 0. Moreover, in virtue of **Exercise 3.6** (Conditions for a unique optimum), our optimal solution is unique since the reduced cost of every nonbasic variable is positive.

(b)
The dual is

$$\begin{aligned} \text{maximize } & p_1 + 2p_2 \\ \text{subject to } & 2p_1 - 3p_2 \leq 4 \\ & -5p_1 + 4p_2 \leq 5 \\ & p_1, p_2 \leq 0. \end{aligned}$$

Note that $p_1 = 0$ and $p_2 = 0$ is an optimal solution to the dual because it is a dual feasible solution and the cost associated to this feasible solution is 0, the optimal cost of the primal. Consequently, applying **Strong Duality** we have that $(0, 0)$ is an optimal solution to the dual.



You can see in the figure the dual feasible set, and actually you can check that $(0, 0)$ is indeed an optimal solution since when we move the orange line which represents the objective function without leaving the feasible set, $(0, 0)$ is the last point that we "touch". The optimal solution is unique since is attained only at $(0, 0)$.

(c)
The idea in here is to apply a variant of parametric programming that can be used when the vector c is kept fixed but the vector b is replaced by $b+\theta d$, where d is a given vector and θ is a scalar. In this case, the zeroth column of the tableau depends on θ. Whenever θ reaches a value at which some basic variable becomes negative, we apply the dual simplex method in order to recover primal feasibility. So, let's begin. We choose as a basis B the matrix formed by the second and the fourth column of A. In this way, we obtain the initial tableau of the dual simplex algorithm

		x_1	x_2	x_3	x_4
	0	4	0	5	0
$x_2 =$	$1-2\theta$	2	1	-5^*	0
$x_4 =$	$2-3\theta$	-3	0	4	1

Note that the reduced costs are non-negative. This is consistent with the dual simplex algorithm since we start with a dual feasible solution and we work to achieve primal feasibility. If $1-2\theta \geq 0$ and $2-3\theta \geq 0$, we also have a primal feasible solution with the same cost, and optimal solution to both problems have been found. In particular, $g(\theta) = 0$ if $\theta \leq 1/2$ where g is the function that represents the optimal cost as a function of θ. This optimal cost is attained at $(0, 1-2\theta, 0, 2-3\theta)$. On the other hand, if $1/2 < \theta \leq 2/3$, then $x_2 < 0$ and $x_4 \geq 0$. The row associated to x_2 is the pivot row. Note that we marked with * the pivot element. We then perform a change of basis: column A_3 enters the basis and A_2 exits. The new tableau is

		x_1	x_2	x_3	x_4
	$1-2\theta$	6	1	0	0
$x_3 =$	$\frac{-1+2\theta}{5}$	$-\frac{2}{5}$	$-\frac{1}{5}$	1	0
$x_4 =$	$\frac{14}{5}-\frac{23}{5}\theta$	$-\frac{7}{5}^*$	$\frac{4}{5}$	0	1

In this case, if $1/2 < \theta \leq 14/23 < 2/3$, then $x_3 \geq 0$ and $x_4 \geq 0$ which implies that the optimal cost is $g(\theta) = -1+2\theta$ and it's attained at $(0, 0, (-1+2\theta)/5, 14/5 - 23/5\theta)$. If $14/23 < \theta \leq 2/3$, then $x_3 > 0$ but $x_4 < 0$. In this case, A_1 enters the basis and A_4 exits. The new tableau is

		x_1	x_2	x_3	x_4
	$13-\frac{124}{7}\theta$	0	$\frac{31}{7}$	0	$\frac{30}{7}$
$x_3 =$	$-1+\frac{12}{7}\theta$	0	$-\frac{3}{7}$	1	$-\frac{2}{7}$
$x_1 =$	$-2+\frac{23}{7}\theta$	1	$-\frac{4}{7}$	0	$-\frac{5}{7}$

Note that for $14/23 < \theta \leq 2/3$, $x_1 \geq 0$ and $x_3 \geq 0$. Then $g(\theta) = -13 + 124/7\theta$ for $14/23 < \theta \leq 2/3$ and it's attained at $(-2+23/7\theta, 0, -1+12/7\theta, 0)$. Let us now go back to the initial tableau and suppose that $\theta > 2/3$. In this case something interesting happens, the pivot element is -5 and we apply the dual simplex algorithm. We obtain the second tableau which is the same as the second tableau that we obtained before. For this second tableau and $\theta > 2/3$, $x_4 < 0$. So, we apply the dual simplex algorithm again and we obtain the third tableau which is the same as the third tableau that we obtained before. For this tableau and $\theta > 2/3$ the zero column is non-negative. Consequently, $g(\theta) = -13 + 124/7\theta$ for $\theta > 2/3$, and we're done. The interesting part that I was talking about is that for this last case $\theta > 2/3$ we did nothing new. This implies that this case could be analyze with the rest of the cases. ■

Problem 5.15 0.5

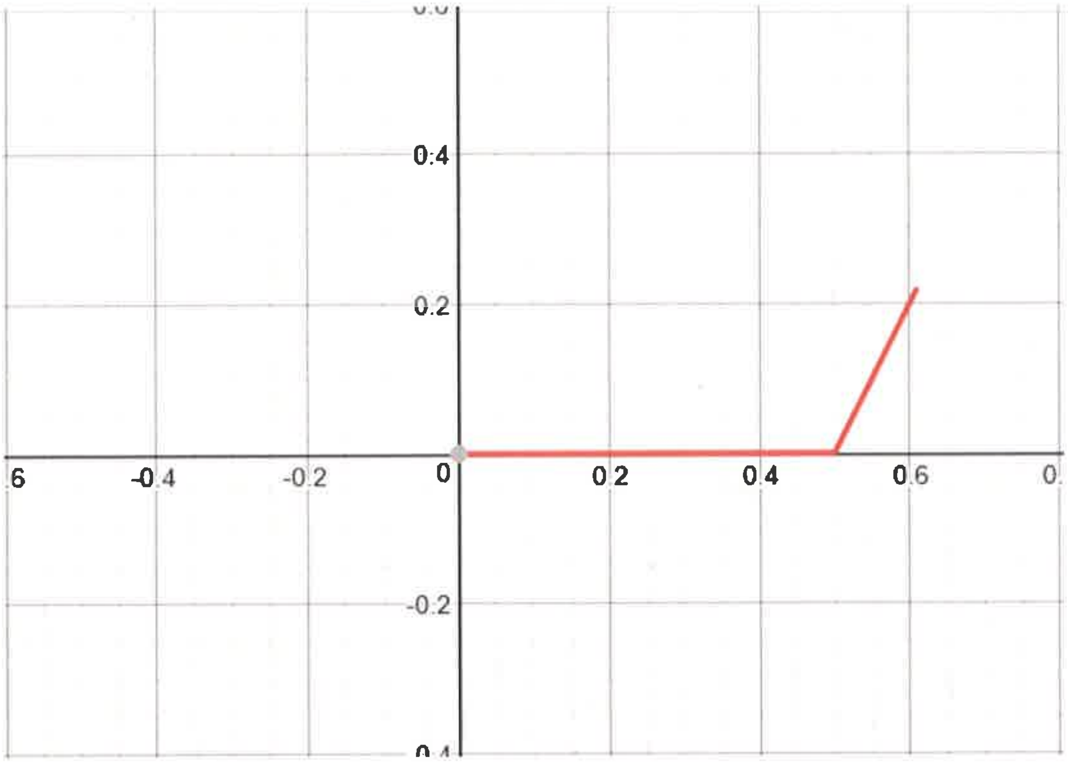
(a)
In **Exercise 5.13** incise c, we had a similar situation. We wanted to find the value of the optimal cost as a function of θ. Recall that the problem was

$$\begin{aligned} \text{minimize } & 4x_1 + 5x_3 \\ \text{subject to } & 2x_1 + x_2 - 5x_3 = 1 - 2\theta \\ & -3x_1 + 4x_3 + x_4 = 2 - 3\theta \\ & x_1, x_2, x_3, x_4 \leq 0. \end{aligned}$$

The optimal cost as a function of θ is

$$g(\theta) = \begin{cases} 0 & \theta \leq \frac{1}{2} \\ -1 + 2\theta & \frac{1}{2} < \theta \leq \frac{14}{23} \\ -13 + \frac{124}{7}\theta & \frac{14}{23} < \theta \end{cases}$$

For this particular case, consider $X(0, 13/24)$ (next figure).



In this case, $X(0, 13/24)$ is not convex, and we're done.

(b)
Assume, by way of contradiction, that when we remove the non-negativity constraints $x \geq 0$ from the previous problem $X(0, 13/24)$ is a convex set. Since the intersection of two convex set is a convex set (**Theorem 2.1**), $X(0, 13/24)$ is also a convex set when we impose again the non-negativity constraint. This is a contradiction because we proved otherwise in incise a. Therefore, our hypothesis is untenable.

(c)
Let $f(\theta) = F(b^* + \theta d)$ where b^* and d are fixed vectors, F as in the book page 214, and θ is a scalar(i.e. f represents the optimal cost as a function of the scalar parameter θ). In the book, they proved that $f(\theta) = \max_{i=1,\dots,n}(p^i)'(b^* + \theta d)$ where $p^1, \dots, p^N$ are the extreme points of the dual feasible set (We are assuming that the matrix A has linearly independent rows). Consequently, f is a continuous function since f is piecewise linear. On the other hand, we have the function $X(\theta)$ which is also a continuous function. $X(0, t) = X(f(\theta))$ where $0 \leq \theta \leq t$. Since the composition of continuous function is continuous and

$[0,t]$ is path-connected, $X(0,t)$ is also path-connected (continuous image of a path-connected space is path-connected). ∎