Score: 8.5

MATH 170 HW#6

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Exercise 4.21 Suppose one of the problem is feasible. By duality, we have two possibilities:

1. the feasible problem is unbounded and the other problem is infeasible
2. Both problem have a finite optimum

In case (1), clearly the set of feasible solutions to the feasible problem is unbounded. Now suppose we are in case (2). Consider the following dual pair.

$$\begin{array}{llcll} \text{minimize} & \sum_{i=1}^{n} -x_i & \qquad & \text{maximize} & \mathbf{0}^{\mathbf{T}}\mathbf{y} \\ \text{subject to} & \mathbf{Ax} \geq \mathbf{0} & & \text{subject to} & \mathbf{y}^{\mathbf{T}}\mathbf{A} \leq (-1, \dots, -1) \\ & \mathbf{x} \geq \mathbf{0} & & & \mathbf{y} \geq \mathbf{0} \end{array}$$

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If the right problem is feasible, then let $\mathbf{y_0}$ be a feasible solution to the right problem. Note $\mathbf{y_0} \neq \mathbf{0}$. Let $\mathbf{p}$ be a solution to the original dual problem. For all $\lambda > 0$, We have $\mathbf{p^T A} + \lambda \mathbf{y_0^T A} \leq \mathbf{c^T} + \lambda(-1, \dots, -1) \leq \mathbf{c^T}$ and $\mathbf{p^T} + \lambda \mathbf{y_0^T} \geq 0$. Therefore, the feasible set of the original dual is unbounded. If the right problem is infeasible, then the left problem is unbounded or infeasible. Since $\mathbf{x} = \mathbf{0}$ is a feasible solution to the left problem, the left problem is unbounded. Let $\hat{\mathbf{x}}$ be a feasible solution to the left problem such that the corresponding objective cost is negative. Note $\hat{\mathbf{x}} \neq \mathbf{0}$. Let $\mathbf{x}$ be a feasible solution to the original primal. For all $\lambda > 0$, we have $\mathbf{Ax} + \lambda \mathbf{A}\hat{\mathbf{x}} \geq \mathbf{b} + \mathbf{0} \geq \mathbf{b}$ and $\mathbf{x} + \lambda \hat{\mathbf{x}} \geq 0$. Therefore, the feasible set of the original primal is unbounded. We conclude that at least one of the original problems is unbounded.

Exercise 4.24 We will first show that every column of the tableau, other than the zeroth column, remains lexicographically positive through out the algorithm. Let the l th row be the pivot row and the j th column be the entering column. By the lexicographic rule, we have

$$\begin{aligned} & \frac{1}{|v_{lj}|} j \text{ th column} \overset{L}{\leq} \frac{1}{|v_{li}|} i \text{ th column} \quad \forall i \neq j,\, v_{li} < 0 \\ \Leftrightarrow \quad & \frac{v_{li}}{v_{lj}} j \text{ th column} \overset{L}{\leq} i \text{ th column} \quad \forall i \neq j,\, v_{li} < 0 \end{aligned} \tag{1}$$

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For the i th column ($i \neq j$, $v_{li} < 0$), after one iteration, the k th entry ($k \neq l$) becomes $v_{ki} - \frac{v_{kj}}{v_{lj}} v_{li} = v_{ki} - \frac{v_{li}}{v_{lj}} v_{kj}$, and the l th entry becomes $\frac{v_{li}}{v_{lj}} > 0$. Therefore, by (1), all the column with the l th entry less than zero remains lexicographically positive. For the i th column ($i \neq j$, $v_{li} \geq 0$), after one iteration, the first entry (the reduced cost of the i th variable) becomes $\bar{c}_i - \frac{\bar{c}_j}{v_{lj}} v_{li} \geq 0$ (as v_{li}, $\bar{c}_j \geq 0$), meaning that it is lexicographically positive. The j th column clearly remains lexicographically positive. We now show that the zero th column strictly decreases lexicographically at each iteration. Similarly, after one iteration, the zero th column is added $-\frac{x_{B(l)}}{v_{lj}}$(j th column$+\mathbf{e}_l$) to it. Since the j th column and $\mathbf{e_j}$ are lexicographically positive and $-\frac{x_{B(l)}}{v_{lj}} < 0$, the zero th column decreases lexicographically at each iteration. Finally, since every basic feasible solution 1-1 corresponds to one zero th column, the dual simplex method will never go back to the same basic feasible solution and hence terminates in finite steps.

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maybe a bit more detail.

Exercise 4.25 The dual problem is equivalent to the following standard form problem.

$$\begin{aligned} \text{minimize} \quad & -p_1 - p_2 \\ \text{subject to} \quad & p_1 + s_1 = 1 \\ & p_2 + s_2 = 1 \\ & p_1, p_2, s_1, s_2 \geq 0 \end{aligned}$$

If the initial basic feasible solution is $p_1 = 0, p_2 = 0, s_1 = 1, s_2 = 1$. Then clearly we need twice changes of basis to obtain our optimal solution $p_1 = 1, p_2 = 1, s_1 = 0, s_2 = 0$.

Exercise 4.29 Consider the following dual pair.

$$\begin{aligned} \text{minimize} \quad & \mathbf{0}^{\mathbf{T}}\mathbf{x} \\ \text{subject to} \quad & \mathbf{A}\mathbf{x} \geq \mathbf{b} \end{aligned} \qquad\qquad \begin{aligned} \text{maximize} \quad & \mathbf{p}^{\mathbf{T}}\mathbf{b} \\ \text{subject to} \quad & \mathbf{p}^{\mathbf{T}}\mathbf{A} = \mathbf{0}^{\mathbf{T}} \\ & \mathbf{p} \geq \mathbf{0} \end{aligned}$$

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Since the system of inequalities $\mathbf{Ax} \geq \mathbf{b}$ is inconsistent, the primal is infeasible. Note that $\mathbf{p} = \mathbf{0}$ is a feasible solution to the dual. Thus, the dual is unbounded. Then there is a basic feasible solution $\hat{\mathbf{p}}$ to the polyhedron $P = \{\mathbf{p} \in \mathbb{R}^m | \, \mathbf{A}^{\mathbf{T}}\mathbf{p} = \mathbf{0}, \, \mathbf{b}^{\mathbf{T}}\mathbf{p} = 1, \, \mathbf{p} \geq \mathbf{0}\}$. Since there are $n+1$ equality constraints in P, at most $n+1$ entries of $\hat{\mathbf{p}}$ are nonzero. Let I be the index set of the nonzero entries. We claim that the subsystem $\mathbf{a}_i^{\mathbf{T}}\mathbf{x} \geq b_i \; \forall i \in I$ is inconsistent. Since $\hat{\mathbf{p}}^{\mathbf{T}}\mathbf{A} = \sum_{i \in I} p_i \mathbf{a}_i^{\mathbf{T}} = \mathbf{0}^{\mathbf{T}}$, $\hat{\mathbf{p}}^{\mathbf{T}}\mathbf{b} = \sum_{i \in I} p_i b_i = 1$ and $\mathbf{p} \geq \mathbf{0}$, by Theorem 4.7, if $\hat{\mathbf{x}}$ is a solution to the subsystem, then we must have $\mathbf{0}^{\mathbf{T}}\mathbf{x} = 0 \leq -1$. Therefore, the subsystem is inconsistent. If the subsystem consists exactly $n+1$ constraints, we are done. Otherwise, choose any constraints into the system to make it consist $n+1$ constraints. Clearly, the resulting system is still inconsistent.

Exercise 4.30

(a) Every polyhedron in $\mathbb{R}^n$ can be regarded as the solution set of some system of linear inequalities $\mathbf{a}_i^{\mathbf{T}}\mathbf{x} \geq \mathbf{b}_i$. Thus, the family of polyhedron $\mathcal{F}$ is related to a bunch of linear inequalities $\mathbf{a}_i^{\mathbf{T}}\mathbf{x} \geq \mathbf{b}_i$, $i = 1, \ldots, m$. Since every $n+1$ polyhedra in $\mathcal{F}$ has a point in common and each polyhedron associates with at least one inequality, any choice of $n+1$ inequalities of the total m inequalities must be consistent. By the contrapositive of Exercise 4.29, the original system of m linear inequalities is consistent, which means all polyhedra in $\mathcal{F}$ have a point in common. 1

(b) No, it is not true. Consider the following polyhedra $P_1 = \{(x,y) \in \mathbb{R}^2 | \, x = 0\}, P_2 = \{(x,y) \in \mathbb{R}^2 | \, y = 0\}, P_3 = \{(x,y) \in \mathbb{R}^2 | \, x + y = 1\}$. Clearly, any two of them have a point in common, but their intersection is empty.

Exercise 4.31 Consider the pair of dual problems:

$$\begin{aligned} \text{minimize} \quad & \mathbf{0}^{\mathbf{T}}\mathbf{x} \\ \text{subject to} \quad & \sum_{k=1}^{n} p_{ik}x_k - x_i \geq 1 \quad \forall i = 1, \ldots, n \end{aligned} \qquad\qquad \begin{aligned} \text{maximize} \quad & \sum_{i=1}^{n} p_i \\ \text{subject to} \quad & \mathbf{p}^{\mathbf{T}}(\mathbf{P} - \mathbf{I}) = \mathbf{0}^{\mathbf{T}} \\ & \mathbf{p} \geq \mathbf{0} \end{aligned}$$

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The left problem seems trivial, but actually it is trivial iff it is feasible. The right problem is feasible because $\mathbf{p} = \mathbf{0}$ is a feasible solution. By duality, if the left problem is infeasible, then the right problem should be unbounded, which means there exists a nonzero feasible solution. We now prove the left problem is infeasible. Assume otherwise there is a feasible solution to the left problem, say $\mathbf{x}$. Let x_j be the largest component of $\mathbf{x}$. Then we have $\sum_{k=1}^{n} p_{jk}x_k - x_j \geq 1 \;\Leftrightarrow\; \sum_{k=1}^{n} p_{jk}x_k \geq 1 + x_j$ However, since $\sum_{k=1}^{n} p_{jk} = 1$ and $p_{jk} \geq 0 \; \forall k$, we also have $\sum_{k=1}^{n} p_{jk}x_k \leq x_j$. Contradiction. We conclude that the left problem is infeasible and hence there exists a nonzero solution to the system $\mathbf{p}^{\mathbf{T}}\mathbf{P} = \mathbf{p}^{\mathbf{T}}$, $\mathbf{p} \geq \mathbf{0}$.

Exercise 4.35

(a) Consider the LP

$$\begin{aligned} \text{minimize} \quad & \mathbf{0}^{\mathbf{T}}(\mathbf{x}+\mathbf{y}) \\ \text{subject to} \quad & \mathbf{A}\mathbf{x} \le \mathbf{b} \\ & \mathbf{D}\mathbf{y} \le \mathbf{d} \\ & \mathbf{x} = \mathbf{y} \end{aligned}$$

If $P \cap Q \neq \emptyset$, then this LP is feasible. By the two-phase simplex method, we will obtain a feasible solution $(\mathbf{x}, \mathbf{y})$, and $\mathbf{x}$ is a point in $P \cap Q$. If $P \cap Q = \emptyset$, clearly this LP is infeasible.

(b) Consider the dual of the above LP

$$\begin{aligned} \text{maximize} \quad & \mathbf{p}_1^{\mathbf{T}}\mathbf{b} + \mathbf{p}_2^{\mathbf{T}}\mathbf{d} \\ \text{subject to} \quad & \mathbf{p}_1^{\mathbf{T}}\mathbf{A} + \mathbf{p}_2^{\mathbf{T}}\mathbf{D} = \mathbf{0}^{\mathbf{T}} \\ & \mathbf{p}_1, \mathbf{p}_2 \le \mathbf{0} \end{aligned}$$

If $P \cap Q = \emptyset$, then the primal is infeasible. The dual is feasible because $\mathbf{p}_1 = \mathbf{p}_2 = \mathbf{0}$ is a feasible solution. By duality, the dual should be unbounded. Then there are vectors $\mathbf{p}_1, \mathbf{p}_2$ such that $\mathbf{p}_1^{\mathbf{T}}\mathbf{b} + \mathbf{p}_2^{\mathbf{T}}\mathbf{d} > 0$, $-\mathbf{p}_1^{\mathbf{T}}\mathbf{A} = \mathbf{p}_2^{\mathbf{T}}\mathbf{D}$, and $\mathbf{p}_1, \mathbf{p}_2 \le \mathbf{0}^{\mathbf{T}}$. We claim that $\mathbf{c}^{\mathbf{T}} = -\mathbf{p}_1^{\mathbf{T}}\mathbf{A} = \mathbf{p}_2^{\mathbf{T}}\mathbf{D}$ has the desired property. Let $\mathbf{x} \in P$, $\mathbf{y} \in Q$.

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$$\mathbf{p}_1, \mathbf{p}_2 \le \mathbf{0}^{\mathbf{T}} \Rightarrow \begin{cases} -\mathbf{p}_1^{\mathbf{T}}\mathbf{A}\mathbf{x} \le -\mathbf{p}_1^{\mathbf{T}}\mathbf{b} \\ \mathbf{p}_2^{\mathbf{T}}\mathbf{D}\mathbf{y} \ge \mathbf{p}_2^{\mathbf{T}}\mathbf{d} \end{cases} \Rightarrow \begin{cases} \mathbf{c}^{\mathbf{T}}\mathbf{x} \le -\mathbf{p}_1^{\mathbf{T}}\mathbf{b} \\ \mathbf{c}^{\mathbf{T}}\mathbf{y} \ge \mathbf{p}_2^{\mathbf{T}}\mathbf{d} \end{cases}$$

$$\mathbf{p}_1^{\mathbf{T}}\mathbf{b} + \mathbf{p}_2^{\mathbf{T}}\mathbf{d} > 0 \Leftrightarrow \mathbf{p}_2^{\mathbf{T}}\mathbf{d} > -\mathbf{p}_1^{\mathbf{T}}\mathbf{b}$$

Combine all inequalities, we have $\mathbf{c}^{\mathbf{T}}\mathbf{y} > \mathbf{c}^{\mathbf{T}}\mathbf{x}$, as desired.

Exercise 4.36

(a) Let $P = \{\mathbf{x} \in \mathbb{R}^n | \, \mathbf{A}\mathbf{x} \le \mathbf{b}\}$ where $\mathbf{A} \in M_{m \times n}$ and $Q = \{\mathbf{x} \in \mathbb{R}^n | \, \mathbf{c}_i^{\mathbf{T}}\mathbf{x} \le d_i \, \forall i = 1, \ldots, l\}$. $P \subseteq Q$ if and only if $\mathbf{A}\mathbf{x} \le \mathbf{b} \Rightarrow \mathbf{c}_i^{\mathbf{T}}\mathbf{x} \le d_i \, \forall i = 1, \ldots, l$. By Theorem 4.7, this is equivalent to saying that $\exists \mathbf{p}_1, \ldots, \mathbf{p}_l \ge \mathbf{0}$ such that $\mathbf{p}_i^{\mathbf{T}}\mathbf{A} = \mathbf{c}_i^{\mathbf{T}}$ and $\mathbf{p}_i^{\mathbf{T}}\mathbf{b} \le d_i$ for all $i = 1, \ldots, l$. Beginning with $i = 1$, we solve the following standard form LP

$$\begin{aligned} \text{minimize} \quad & \mathbf{b}^{\mathbf{T}}\mathbf{p} \\ \text{subject to} \quad & \mathbf{A}^{\mathbf{T}}\mathbf{p} = \mathbf{c}_i \\ & \mathbf{p} \ge \mathbf{0} \end{aligned}$$

If at some i the LP is infeasible or has optimal cost greater than d_i, then we conclude that $P \not\subseteq Q$. Otherwise, $P \subseteq Q$.

(b) Let

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$$P = \left\{ \sum_{i=1}^{k} \lambda_i \mathbf{x}_i^P + \sum_{j=1}^{r} \theta_j \mathbf{w}_j^P \,|\, \lambda_i \ge 0, \, \theta_j \ge 0, \, \sum_{i=1}^{k} \lambda_i = 1 \right\}$$

$$Q = \left\{ \sum_{i=1}^{m} \lambda_i \mathbf{x}_i^Q + \sum_{j=1}^{n} \theta_j \mathbf{w}_j^Q \,|\, \lambda_i \ge 0, \, \theta_j \ge 0, \, \sum_{i=1}^{m} \lambda_i = 1 \right\}$$

If P is bounded, then P has no extreme ray and is the convex hull of its extreme points. In this case, we can simply check if $\mathbf{x}_i^P \in Q$ for all i. We prove this algorithm works now.

Suppose $\mathbf{x}_i^P \in Q \quad \forall i$. Let $p \in P$. Then

$$p = \sum_{i=1}^{k} \lambda_i \mathbf{x}_i^P = \sum_{i=1}^{k} \lambda_i \left(\sum_{a=1}^{m} \lambda_a^i \mathbf{x}_a^Q + \sum_{b=1}^{n} \theta_b^i \mathbf{w}_b^Q \right)$$
$$= \sum_{a=1}^{m} \left(\sum_{i=1}^{k} \lambda_i \lambda_a^i \right) \mathbf{x}_a^Q + \sum_{b=1}^{n} \left(\sum_{i=1}^{k} \lambda_i \theta_b^i \right) \mathbf{w}_b^Q$$

Note all the coefficients are nonnegative and $\sum_{a=1}^{m} \sum_{i=1}^{k} \lambda_i \lambda_a^i = 1$. Thus, $P \subseteq Q$. Now suppose P is unbounded. If Q is bounded, then clearly $P \not\subseteq Q$. If Q is also unbounded, then we check whether $\mathbf{x}_i^P \in Q$ for all i and $\mathbf{w}_j^Q$ can be expressed as a positive linear combination of the extreme rays of Q for all j. A similar argument shows this algorithm works too.

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Exercise 4.39 Let $C = \{\mathbf{x} \in \mathbb{R}^n | \mathbf{a}_i^\mathbf{T}\mathbf{x} \geq \mathbf{0} \ \forall i = 1, \ldots, m\}$. Since C is pointed, there are n linearly independent constraints among $\mathbf{a}_i^\mathbf{T}\mathbf{x} \geq \mathbf{0}$'s and $m \geq n$. Let $\mathbf{d} \in C$ be nonzero. Suppose there are $n-1$ linearly independent constraints active at $\mathbf{d}$ and the index set of these constraints is I. Let $\mathbf{f}, \mathbf{g} \in C$ such that $\mathbf{d} = \mathbf{f} + \mathbf{g}$. Then $\mathbf{a}_i^\mathbf{T}\mathbf{d} = \mathbf{a}_i^\mathbf{T}\mathbf{f} + \mathbf{a}_i^\mathbf{T}\mathbf{g} = 0$ for all $i \in I$. Because $\mathbf{f}, \mathbf{g} \in C$, $\mathbf{a}_i^\mathbf{T}\mathbf{f}, \mathbf{a}_i^\mathbf{T}\mathbf{g} \geq 0 \ \forall i \in I$. Therefore, $\mathbf{a}_i^\mathbf{T}\mathbf{f} = \mathbf{a}_i^\mathbf{T}\mathbf{g} = 0 \ \forall i \in I$. That said, $\mathbf{f}, \mathbf{g}, \mathbf{d}$ are all in the kernal of the linear system $\mathbf{a}_i^\mathbf{T}\mathbf{x} = 0 \ \forall i \in I$. Since the nullity of the linear system is 1, $\mathbf{f}, \mathbf{g}$ are scalar multiples of $\mathbf{d}$. Now suppose $\mathbf{d}$ has the property in the second definition. Choose $\mathbf{c} \in \mathbb{R}^n$ such that $\mathbf{c}^\mathbf{T}\mathbf{d} = 1$. Consider the polyhedron $P = \{\mathbf{x} \in \mathbb{R}^n | \mathbf{a}_i^\mathbf{T}\mathbf{x} \geq \mathbf{0} \ \forall i = 1, \ldots, m \ , \ \mathbf{c}^\mathbf{T}\mathbf{x} = 1\}$. We claim that $\mathbf{d}$ is an extreme point of P. Assume otherwise $\mathbf{d}$ is not an extreme point of P. Then there exist $\mathbf{f}, \mathbf{g} \in P \backslash \{\mathbf{d}\}, \lambda \in (0, 1)$ such that $\mathbf{d} = \lambda\mathbf{f} + (1 - \lambda)\mathbf{g}$. Note $\lambda\mathbf{f}, (1 - \lambda)\mathbf{g} \in C$. Hence, by our assumption, there are scalars $k_1, k_2 \neq 0$ such that $\lambda\mathbf{f} = k_1\mathbf{d}$, $(1 - \lambda)\mathbf{g} = k_2\mathbf{d}$. Multiplying $\mathbf{c}^\mathbf{T}$ to both sides yields $\lambda = k_1, (1 - \lambda) = k_2$. That said, $\mathbf{f} = \mathbf{g} = \mathbf{d}$. Contradiction. Thus, $\mathbf{d}$ is an extreme point of P. Then there are n linearly independent constraints of P active at $\mathbf{d}$. By our construction of P, there are $n - 1$ linearly independent constraints of C active at $\mathbf{d}$, as desired.